ON THE UNDECIDABILITY OF POWER SERIES FIELDS

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Let $F$ be a field and $F((T))$ the field of formal power series over $F$.

**Theorem.** If $F$ is undecidable, then $F((T))$ is undecidable.

**Remark.** Malcev [1] obtained special cases of this result.

**Proof.** It suffices to show that the valuation subring $A = F[[T]]$ is elementarily definable in $F((T))$ [3, Sec. 6]. Fix $m > 1$ such that $\text{char}(F) | m$. An idea in J. Robinson [3] shows that $A$ is definable in terms of $T$: $A = \{x \mid \exists y [y^m = 1 +Tx^m]\}$. By compounding this idea and another trick we can get rid of $T$: $A = \{x \mid \exists w \forall y [w \forall x_1 \forall x_2 \exists z \forall y_1 \forall y_2 [(x^m = 1 + wx_1^m \vee y_1^m \neq 1 + wx_2^m) \wedge u^m \neq w \wedge y^m = 1 + wx^m]\}$. Indeed, this follows from the fact that $A = \bigcup_{w \in G} A_w$, where $A_w = \{x \mid \exists y [y^m = 1 + wx^m]\}$, if $G \subset F((T))$ has the following properties: (1) $T \in G$; (2) for each $w \in G$, $A_w$ is closed under multiplication and its elements have poles of bounded order. (1) shows that $\bigcup_{w \in G} A_w \supseteq A$ while (2) gives the reverse inclusion.

**Corollary.** If $\text{char}(F) = 0$, then $F$ is decidable if and only if $F((T))$ is decidable.

**Proof.** Combine the above theorem with Theorem 6 of Ax-Kochen [4].

Finally, we note that the theorem, the corollary and their proofs remain valid if $F((T))$ is replaced by any Hensel field valued in a $Z$-group with residue class field $F$.

**References**


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