1. Introduction. Let $S$ denote the family of functions $f$ which are regular and univalent in the unit disk $|z| < 1$, hereafter called $\Delta$, and which satisfy the conditions $f(0) = 0$ and $f'(0) = 1$, and let $S^*$, $K$ and $C$ be the subfamilies of $S$ whose members map $\Delta$ onto domains which are starlike with respect to the origin, convex, and close-to-convex, respectively. Then, as was shown by W. Kaplan [2],

\begin{equation}
\mathcal{K} \subset S^* \subset C \subset S.
\end{equation}

Recently, in a seminar given at Rutgers University, Professor M. S. Robertson showed that the starlike function $k$, where $k(z) = z(1 - z)^{-2}$, has the property that $2z^{-1}\int_0^z k(t) \, dt$, $z \in \Delta$, defines a function in $S^*$. The extremal character of the Koebe function, $k$, within the class $S^*$, suggests the following generalization.

**Theorem 1.** If $s$ is in $S^*$, then the function $S$, defined by $S(z) = (2/z)\int_0^z s(t) \, dt$, is likewise in $S^*$.

It is the purpose of this note to establish Theorem 1 and to consider similar conclusions for other members of $S$.

2. Preliminary results. The class of all regular functions $P$ which satisfy the conditions $P(0) = 1$ and $\text{Re}\{P(z)\} > 0$, for $z$ in $\Delta$, is represented by $\mathcal{P}$.

**Lemma 1.** If $N$ and $D$ are regular in $\Delta$, $N(0) = P(0) = 0$, $D$ maps $\Delta$ onto a many-sheeted region which is starlike with respect to the origin, and $N'/D' \in \mathcal{P}$, then $N/D \in \mathcal{P}$.

**Remark.** The essential ideas in the proof of Lemma 1 are the same as given by Sakaguchi in the case $D$ is univalent [6]. R. M. Robinson [5, Lemma, p. 30] has used a similar technique.

**Proof.** By known properties of class $\mathcal{P}$, [4], we can write

\begin{equation}
\left| \frac{N'(z)}{D'(z)} - a(r) \right| < a(r), \quad |z| < r, \quad 0 < r < 1.
\end{equation}

Choose $A(z)$ so that

\begin{equation}
D'(z)A(z) = N'(z) - a(r)D'(z) \quad \text{and} \quad |A(z)| < a(r),
\end{equation}

Received by the editors December 30, 1963 and, in revised form, March 27, 1964.
for $|z| < r$. Fix $z_0, \bar{z_0} \in \Delta$, and let $L$ be the segment joining 0 to $D(z_0)$ which lies in one sheet of the starlike image of $\Delta$ by the mapping $D$. Let $L^{-1}$ be the pre-image of $L$ under $D$ and let $r = \max |z|$, where $z \in L^{-1}$. Then

$$|N(z_0) - a(r)D(z_0)| = \left| \int_0^{z_0} \left[ N'(t) - a(r)D'(t) \right] dt \right|$$

$$= \left| \int_{L^{-1}} D'(t)A(t) dt \right| \leq a(r) \int_{L^{-1}} |dD(t)|$$

$$= a(r) |D(z_0)|.$$

This proves the lemma.

**Lemma 2.** If $s \in \mathbb{S}^*$, then $\sigma(z) = \int_0^s t \, dt$, $z \in \Delta$, gives a function which is 2-valently starlike with respect to the origin for all $z$ in $\Delta$.

**Proof.** Let $D(z) = z\sigma'(z) = z\bar{s}(z)$ and $N(z) = a(z)$, then $D$ is (2-valently) starlike with respect to the origin since

$$\text{Re} \left\{ \frac{zD'(z)}{D(z)} \right\} = \text{Re} \left\{ 1 + \frac{zs'(z)}{s(z)} \right\} > 1 > 0 \quad \text{for} \quad z \in \Delta.$$

Furthermore,

$$\text{Re} \left\{ \frac{N'(z)}{D'(z)} \right\} > 0, \quad z \in \Delta,$$

because

$$\text{Re} \left\{ \frac{D'(z)}{N'(z)} \right\} = \text{Re} \left\{ 1 + \frac{zs'(z)}{s(z)} \right\} > 0.$$

An application of Lemma 1 [which is valid even though $N'(0)/D'(0) \neq 1$] yields

$$\text{Re} \left\{ \frac{N(z)}{D(z)} \right\} > 0, \quad \text{or} \quad \text{Re} \left\{ \frac{z\sigma'(z)}{\sigma(z)} \right\} > 0, \quad \text{for} \quad z \in \Delta;$$

and this together with

$$\int_0^{2\pi} \text{Re} \left\{ \frac{r e^{i\theta} \sigma'(r e^{i\theta})}{\sigma(r e^{i\theta})} \right\} \, d\theta = 2\pi \left\{ \frac{z\sigma'(z)}{\sigma(z)} \right\}_{z=0} = 4\pi, \quad 0 < r < 1,$$

which follows from the mean-value theorem for harmonic functions, shows that $\sigma$ is 2-valent and starlike [1, p. 212]. One can show, furthermore, that $\sigma$ is convex [1] throughout the unit disk.
3. Theorems and proofs.

Proof of Theorem 1. For the function $S$, defined in Theorem 1, we have

\[
\frac{zS'(z)}{S(z)} = \frac{zs(z) - \int_0^z s(t) \, dt}{\int_0^z s(t) \, dt} = \frac{zo'(z) - \sigma(z)}{\sigma(z)}
\]

and then differentiation of the numerator and the denominator of the last expression gives

\[
\frac{[zo'(z) - \sigma(z)]'}{\sigma'(z)} = \frac{zo''(z)}{\sigma'(z)} = \frac{zs'(z)}{S(z)}.
\]

An application of Lemma 1 and Lemma 2 completes the proof.

As an immediate consequence of Theorem 1 and the fact that

\[(3.1) \quad f \in \mathcal{K} \quad \text{if and only if} \quad zf'' \in \mathcal{S}^*,
\]

we have the following corollary.

**Corollary 1.1.** \(s \in \mathcal{S}^*\), then \(f(r(t))\) \([f0's(x)dx\] \(dt\) defines a member of \(\mathcal{K}\).

**Theorem 2.** If \(c \in \mathcal{K}\) and \(C(z) = (2/z)\int_0^z c(t) \, dt\), then \(C \in \mathcal{K}\).

**Proof.** Let \(s(z) = zc'(z)\); then \([by (3.1)]\) \(s \in \mathcal{S}^*\). Let \(S\) be defined as in Theorem 1, then

\[
S(z) = \frac{2}{z} \int_0^z tc'(t) \, dt = \frac{2}{z} \left[ zc(z) - \int_0^z c(t) \, dt \right]
\]

\[
= z \left[ \frac{2(z)}{z} - \frac{2}{z} \int_0^z c(t) \, dt \right] = zC'(z).
\]

Since \(S\) is starlike, it follows from (3.1) that \(C\) is convex. Theorem 2 can be proved directly from Lemma 1 by the method of Theorem 1.

**Corollary 2.1.** If \(c \in \mathcal{K}\), \(C(z) = (2/z)\int_0^z c(t) \, dt\) and \(h(z) = 2c(z) - C(z)\), then \(h \in \mathcal{S}^*\).

\(f\) is in \(\mathcal{C}\) if, and only if, there exists a function \(g\) such that \(g(z) = \epsilon s(z)\), \(s \in \mathcal{S}^*, \|\epsilon\| = 1\), and

\[
\text{Re} \left\{ \frac{(zf'(z))}{g(z)} \right\} > 0, \quad \text{for } z \in \Delta.
\]
In this case, we say \( f \) is close-to-convex with respect to the (starlike) function \( g \).

**Theorem 3.** If \( f \) is close-to-convex with respect to \( g \),
\[
F(z) = \frac{2}{z} \int_0^z f(t) \, dt \quad \text{and} \quad G(z) = \frac{2}{z} \int_0^z g(t) \, dt,
\]
then \( F \) is close-to-convex with respect to \( G \).

The proof is similar to that of Theorem 1.

In [3], J. Krzyż showed that the radius of close-to-convexity of every function in \( S \) is greater than or equal to \( r_0, .80 < r_0 < .81 \).

**Theorem 4.** If \( f \in S \) and \( F(z) = (2/z) \int_0^z f(t) \, dt \), then \( F \) is schlicht (and close-to-convex) for \( |z| < r, r \geq r_0 \).

**Acknowledgment.** The author thanks Professor Thomas H. McGregor for his valuable suggestions and criticisms.

**References**


**University of Delaware**