PARTIALLY ISOMETRIC TOEPLITZ OPERATORS

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It has been observed that the only Toeplitz operators that are isometric are those of the form $T_\phi$ where $\phi$ is an inner function on the unit disc (see e.g., [1, Theorem 8, Corollary 3]; the notation and terminology introduced there will be adhered to throughout this note). If we turn to the somewhat more general question of which Toeplitz operators are partial isometries, we see at once that the isometries $T_\phi$, as well as their adjoints $T_\phi^*$, are such, and a hasty inventory leads to the guess that there are no others. The purpose of this note is to prove that this is, in fact, the case.

Theorem. The only Toeplitz operators (other than the operator 0) that are partial isometries are those of the form $T_\phi$ and $T_\phi^*$, where $\phi$ is an inner function.

The argument depends on the following pair of lemmas which are perhaps, of some interest in their own right.

Lemma 1. Let $V$ be an isometry on a Hilbert space $\mathcal{H}$ and let $T$ be a bounded operator that commutes $V$. Suppose that the deficiency of $V$, i.e., the subspace $\mathcal{N} = (V\mathcal{H})^\perp$, is invariant under $T$. Then $T$ also commutes $V^*$.

Proof. If $x \in \mathcal{N}$ then $TV^*x = 0 = V^*Tx$. Thus $[T, V^*] = TV^* - V^*T$ annihilates $\mathcal{N}$. Suppose, inductively, that $V^k(\mathcal{N})$ is invariant under $T$ and annihilated by $[T, V^*]$. If $x \in V^k(\mathcal{N})$ then $x = Vy$, $y \in V^{k+1}(\mathcal{N})$, and we have:

$$Tx = TVy = VTy \in V^{k+1}(\mathcal{N});$$

$$TV^*x = Ty = V^*VTy = V^*TVy = V^*Tx.$$ 

It follows that $\mathcal{N} = \sum_{k=0}^\infty \oplus V^k(\mathcal{N})$ is also invariant under $T$ and that $TV^* = V^*T$ on vectors belonging to $\mathcal{N}$. But [2, Lemma 2.1] $\mathcal{N}$ is a reducing subspace for $V$ and $V$ induces a unitary operator on $\mathcal{N}^\perp$. It follows at once that $\mathcal{N}^\perp$ is also invariant under $T$, i.e., that $\mathcal{N}^\perp$ reduces $T$ as well as $V$. That $T$, acting on $\mathcal{N}$, commutes $V^*$ has just been verified; that $T$, acting on $\mathcal{N}^\perp$, commutes $V^*$ follows by Fuglede's Theorem, and the proof is complete.

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Lemma 2. Let $T_\psi$ be a Toeplitz operator that achieves its norm. Then $T_\psi = \lambda T_\phi^* T_\phi$, where $\lambda > 0$ and $\phi, \psi$ are inner functions.

Proof. We may suppose that $T_\psi \neq 0$ and hence that $\|T_\psi\| = 1$. Let $G = \{ g \in \mathbb{C}^2 : \|T_\psi g\| = \|g\| \}$ and let $g_0 \in G, g_0 \neq 0$. Since $1 = \|T_\psi\| = \|L_\psi\| = \|f\|$, we have $\|g_0\| = \|T_\psi g_0\| = \|PL_\psi g_0\| \leq \|f g_0\| \leq \|g_0\|$. (Here $P$ denotes, as usual, the projection of $\mathbb{C}^2$ onto $\mathbb{C}^2$. From this it follows, on the one hand, that $P(f g_0) = f g_0$, i.e., that $f g_0 \in \mathbb{C}^2$, and, on the other hand, that $\|f g_0\| = \|g_0\|$. But now, by a well-known theorem of the brothers Riesz, $g_0 \neq 0$ a.e. on the circle, so that $\|f g_0\| = \|g_0\|$ implies $|f| = 1$ a.e. But then $\|f g\| = \|g\|$ for every $g \in \mathbb{C}^2$. Conclusion: if $g \in \mathbb{C}^2$ then $g \in G$ if and only if $f g \in \mathbb{C}^2$.

It is now a simple matter to see that $G$ is a subspace of $\mathbb{C}^2$ invariant under multiplication by $z$. Indeed, (i) if $g_1, g_2 \in G$ then $f(g_1 + g_2) = f g_1 + f g_2 \in \mathbb{C}^2$ so $g_1 + g_2 \in G$ and (ii) if $g \in G$ then $f(z)(g(z)) = z(f(z)g(z)) \in \mathbb{C}^2$. Hence, by the fundamental theorem on such subspaces (see e.g., [3, Theorem 4]), there exists an inner function $\phi$ such that $G = T_\phi(\mathbb{C}^2)$. Moreover, $\phi \in G$ so that $\psi = f \phi \in \mathbb{C}^2$, and since $|\psi| = |f| |\phi| = 1$ a.e., we see that $\psi$ is also an inner function. Finally, we have $f = \phi^{-1} \psi = \phi \psi$ and therefore [1, Theorem 8] $T_\psi = T_\phi^* T_\phi = T_\phi^* T_\psi$.

Proof of the theorem. A partial isometry assumes its norm so the last lemma applies. Adopting the notation of the lemma, suppose that $h \in \mathbb{C}^2 \ominus G$. Since $T_\psi$ is a partial isometry with initial space $G$, we have $T_\psi h = T_\phi^* (T_\psi h) = 0$ and consequently $T_\psi h \in \mathbb{C}^2 \ominus G$. Thus $\mathbb{C}^2 \ominus G$, which is precisely the deficiency of the isometry $T_\phi$, is invariant under $T_\psi$ and by Lemma 1, $T_\psi$ commutes $T_\phi^*$ as well as $T_\phi$. But then, by [1, Theorem 9], it follows at once that either $\phi$ or $\psi$ must be constant.

Added in proof. Lemma 2 may be viewed as providing an answer to a question raised by H. Helson [Invariant subspaces, Academic Press, New York, 1964; p. 12]: a unitary function $\theta$ is the quotient of two inner functions when and only when the Toeplitz operator $T_\theta$ achieves its norm.

References