PARTIALLY ISOMETRIC TOEPLITZ OPERATORS

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It has been observed that the only Toeplitz operators that are isometric are those of the form T_ϕ where ϕ is an inner function on the unit disc (see e.g., [1, Theorem 8, Corollary 3]; the notation and terminology introduced there will be adhered to throughout this note). If we turn to the somewhat more general question of which Toeplitz operators are *partial* isometries, we see at once that the isometries T_ϕ, as well as their adjoints T_ϕ^*, are such, and a hasty inventory leads to the guess that there are no others. The purpose of this note is to prove that this is, in fact, the case.

THEOREM. *The only Toeplitz operators (other than the operator* 0) *that are partial isometries are those of the form T_ϕ and T_ϕ^*, where ϕ is an inner function.*

The argument depends on the following pair of lemmas which are perhaps, of some interest in their own right.

LEMMA 1. *Let V be an isometry on a Hilbert space $\mathfrak{H}$ and let T be a bounded operator that commutes V. Suppose that the deficiency of V, i.e., the subspace $\mathfrak{N}=(V\mathfrak{H})^\perp$, is invariant under T. Then T also commutes V^*.*

PROOF. If $x\in\mathfrak{N}$ then $TV^*x=0=V^*Tx$. Thus $[T, V^*]=TV^* - V^*T$ annihilates $\mathfrak{N}$. Suppose, inductively, that $V^k(\mathfrak{N})$ is invariant under T and annihilated by $[T, V^*]$. If $x\in V^{k+1}(\mathfrak{N})$ then $x=Vy$, $y\in V^k(\mathfrak{N})$, and we have:

$$Tx = TVy = VTy \in V^{k+1}(\mathfrak{N});$$

$$TV^*x = Ty = V^*VTy = V^*TVy = V^*Tx.$$

It follows that $\mathfrak{M}=\sum_{k=0}^\infty \oplus V^k(\mathfrak{N})$ is also invariant under T and that $TV^*=V^*T$ on vectors belonging to $\mathfrak{M}$. But [2, Lemma 2.1] $\mathfrak{M}$ is a reducing subspace for V and V induces a unitary operator on $\mathfrak{M}^\perp$. It follows at once that $\mathfrak{M}^\perp$ is also invariant under T, i.e., that $\mathfrak{M}^\perp$ reduces T as well as V. That T, acting on $\mathfrak{M}$, commutes V^* has just been verified; that T, acting on $\mathfrak{M}^\perp$, commutes V^* follows by Fuglede's Theorem, and the proof is complete.

Received by the editors May 27, 1964.

[1] Research supported in part by the National Science Foundation.

LEMMA 2. *Let T_f be a Toeplitz operator that achieves its norm. Then $T_f = \lambda T_\phi^* T_\psi$ where $\lambda > 0$ and ϕ, ψ are inner functions.*

PROOF. We may suppose that $T_f \neq 0$ and hence that $\|T_f\| = 1$. Let $\mathcal{G} = \{g \in \mathcal{H}^2 : \|T_f g\| = \|g\|\}$ and let $g_0 \in \mathcal{G}$, $g_0 \neq 0$. Since $1 = \|T_f\| = \|L_f\| = \|f\|_\infty$ we have $\|g_0\| = \|T_f g_0\| = \|PL_f g_0\| \leqq \|fg_0\| \leqq \|g_0\|$. (Here P denotes, as usual, the projection of $\mathcal{L}^2$ onto $\mathcal{H}^2$.) From this it follows, on the one hand, that $P(fg_0) = fg_0$, i.e., that $fg_0 \in \mathcal{H}^2$, and, on the other hand, that $\|fg_0\| = \|g_0\|$. But now, by a well-known theorem of the brothers Riesz, $g_0 \neq 0$ a.e. on the circle, so that $\|fg_0\| = \|g_0\|$ implies $|f| = 1$ a.e. But then $\|fg\| = \|g\|$ for every $g \in \mathcal{H}^2$. Conclusion: if $g \in \mathcal{H}^2$ then $g \in \mathcal{G}$ if and only if $fg \in \mathcal{H}^2$.

It is now a simple matter to see that $\mathcal{G}$ is a subspace of $\mathcal{H}^2$ invariant under multiplication by z. Indeed, (i) if g_1, $g_2 \in \mathcal{G}$ then $f(g_1 + g_2) = fg_1 + fg_1 \in \mathcal{H}^2$ so $g_1 + g_2 \in \mathcal{G}$ and (ii) if $g \in \mathcal{G}$ then $f(z)(zg(z)) = z(f(z)g(z)) \in \mathcal{H}^2$. Hence, by the fundamental theorem on such subspaces (see e.g., [**3**, Theorem 4]), there exists an inner function ϕ such that $\mathcal{G} = T_\phi(\mathcal{H}^2)$. Moreover, $\phi \in \mathcal{G}$ so that $\psi = f\phi \in \mathcal{H}^2$, and since $|\psi| = |f|\,|\phi| = 1$ a.e., we see that ψ is also an inner function. Finally, we have $f = \phi^{-1}\psi = \bar{\phi}\psi$ and therefore [**1**, Theorem 8] $T_f = T_{\bar{\phi}} T_\psi = T_\phi^* T_\psi$.

PROOF OF THE THEOREM. A partial isometry assumes its norm so the last lemma applies. Adopting the notation of the lemma, suppose that $h \in \mathcal{H}^2 \ominus \mathcal{G}$. Since T_f is a partial isometry with initial space $\mathcal{G}$, we have $T_f h = T_\phi^*(T_\psi h) = 0$ and consequently $T_\psi h \in \mathcal{H}^2 \ominus \mathcal{G}$. Thus $\mathcal{H}^2 \ominus \mathcal{G}$, which is precisely the deficiency of the isometry T_ϕ, is invariant under T_ψ and by Lemma 1, T_ψ commutes T_ϕ^* as well as T_ϕ. But then, by [**1**, Theorem 9], it follows at once that either ϕ or ψ must be constant.

Added in proof. Lemma 2 may be viewed as providing an answer to a question raised by H. Helson [*Invariant subspaces*, Academic Press, New York, 1964; p. 12]: a unitary function θ is the quotient of two inner functions when and only when the Toeplitz operator T_θ achieves its norm.

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