ORDER CONVERGENCE AND TOPOLOGICAL CONVERGENCE

RALPH E. DEMARR

In a complete lattice it is possible to define a notion of convergence (for arbitrary nets) known as order convergence ($o$-convergence); for definitions see [1, p. 59] and [3, p. 65]. As a general rule $o$-convergence is not a topological convergence; i.e., the lattice cannot be topologized so that nets $o$-converge if and only if they converge with respect to the topology [2]. It is of interest to know when these two types of convergence coincide. A number of questions could be posed here, but we shall deal only with the question of when a topological space can be suitably embedded in a complete lattice. To make this statement more precise we shall give the following definition.

**Definition.** A topological space is said to be an $O$-space if it is homeomorphic to a subset $\Omega_0$ of a complete lattice $\Omega$ and if every net in $\Omega_0$ converges (with respect to the topology for $\Omega_0$) to a limit in $\Omega_0$ if and only if it $o$-converges to this limit. For example, every completely regular Hausdorff space is an $O$-space because it is homeomorphic to a subset of the direct product of unit intervals and if this direct product is partially ordered componentwise, then it becomes a complete lattice in which $o$-convergence is the same as convergence with respect to the product topology. In this paper we prove the following theorem.

**Theorem.** A topological space is an $O$-space if and only if it is a regular Hausdorff space.

**Proof.** If $X$ is an $O$-space, then it is homeomorphic to a subset $\Omega_0$ of a complete lattice $\Omega$, where the topology for $\Omega_0$ has the properties stated in the above definition. Hence, we only need to show that $\Omega_0$ is a regular Hausdorff space. Since limits with respect to $o$-convergence are unique, $\Omega_0$ is a Hausdorff space. For each nonempty open subset $U$ of $\Omega_0$ define $f(U) = \sup \{ \sigma: \sigma \in U \}$ and $g(U) = \inf \{ \sigma: \sigma \in U \}$. Then define $S(U) = \{ \tau: \tau \in \Omega_0 \text{ and } g(U) \leq \tau \leq f(U) \}$. Since $\Omega_0$ is an $O$-space, $S(U)$ is closed. It is clear that $U \subseteq S(U)$.

Now let $\sigma_0 \in \Omega_0$ be any point. We now define the directed set $D$ to be the collection of all pairs $(U, \sigma)$, where $U \subseteq \Omega_0$ is an open neighborhood of $\sigma_0$ and $\sigma \subseteq U$. The binary relation $<$ is defined as follows: $(U_1, \sigma_1) < (U_2, \sigma_2)$ iff $U_2 \subseteq U_1$. We will now construct a net in $\Omega_0$ as

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follows: for each \( n = (U, \sigma) \in D \) define \( \mu_n = \sigma \). It is clear that the net \( \{\mu_n : n \in D\} \) converges to \( \sigma_0 \) with respect to the topology for \( \Omega_0 \), hence, it must also \( \sigma \)-converge to \( \sigma_0 \). From the definition of \( \sigma \)-convergence we see that \( \inf \{f(U) : U \in N(\sigma_0)\} = \sigma_0 = \sup \{g(U) : U \in N(\sigma_0)\} \), where \( N(\sigma_0) \) denotes the collection of all open neighborhoods \( U \subset \Omega_0 \) of \( \sigma_0 \).

We will use this latter fact to show by contradiction that \( \Omega_0 \) is regular. If \( \Omega_0 \) is not regular then there exists a point \( \sigma_0 \in \Omega_0 \) and an open neighborhood \( V_0 \) \((V_0 \subset \Omega_0)\) of \( \sigma_0 \) such that for every open neighborhood \( U \) of \( \sigma_0 \) we have \( S(U) \cap V_0' \neq \emptyset \). \((V_0') \) denotes the complement of \( V_0 \). Therefore, for each \( U \in N(\sigma_0) \) one may select \( h(U) \in S(U) \cap V_0' \). Now \( \{h(U) : U \in N(\sigma_0)\} \) is a net and since \( g(U) \leq h(U) \leq f(U) \) for all \( U \in N(\sigma_0) \), this net must \( \sigma \)-converge to \( \sigma_0 \) (recall the results of the previous paragraph). But since \( h(U) \in V_0' \) for all \( U \in N(\sigma_0) \), this net cannot converge to \( \sigma_0 \) with respect to the topology for \( \Omega_0 \). This contradicts the fact that \( \Omega_0 \) is an \( O \)-space; hence, \( \Omega_0 \) must be regular. This completes the proof that every \( O \)-space is a regular Hausdorff space.

Now assume that \( X \) is a regular Hausdorff space. We may assume that \( X \) contains infinitely many points, otherwise all is trivial. Let \( \mathcal{F} \) be the collection of all closed subsets of \( X \) which contain at least two points. Now define three sets \( \Omega_- \), \( \Omega_0 \), and \( \Omega_+ \) as follows: \( \Omega_- \) and \( \Omega_+ \) are sets of ordered pairs of the form \((-1, E)\) and \((+1, E)\), respectively, where \( E \subset \mathcal{F} \); \( \Omega_0 \) is the set of ordered pairs of the form \((0, x)\), where \( x \in X \). Then define \( \Omega = \Omega_- \cup \Omega_0 \cup \Omega_+ \). The set \( \Omega \) is partially ordered as follows:

\[\begin{align*}
(-1, E) &\leq (-1, F) \quad \text{iff} \quad F \subset E, \\
(-1, E) &\leq (0, x) \quad \text{iff} \quad x \in E, \\
(-1, E) &\leq (+1, F) \quad \text{iff} \quad E \cap F \neq \emptyset, \\
(0, x) &\leq (0, y) \quad \text{iff} \quad x = y, \\
(0, x) &\leq (+1, E) \quad \text{iff} \quad x \in E, \\
(+1, E) &\leq (+1, F) \quad \text{iff} \quad E \subset F. 
\end{align*}\]

It is easily shown that \( \leq \) is indeed a partial ordering. It is clear that \((-1, X)\) and \((+1, X)\) are the smallest and largest elements, respectively, in \( \Omega \).

In general, \( \Omega \) is not a lattice, but it can be embedded in a complete lattice \( \overline{\Omega} \) (the MacNeille completion); see [1, p. 58]. By the embedding, we can regard \( \Omega \) as a subset of \( \overline{\Omega} \). Hence, \( \Omega_0 \) can be regarded as a subset of \( \overline{\Omega} \). We shall topologize \( \Omega_0 \) so that it is homeomorphic to \( X \); this can be done directly since there is a natural one-to-one correspondence between \( X \) and \( \Omega_0 \).
We will now show that a net in \( \Omega_0 \) converges with respect to the topology for \( \Omega_0 \) if and only if it \( \sigma \)-converges. Let \( \{ \sigma_n : n \in D \} \) be a net in \( \Omega_0 \) which converges to \( \sigma \) with respect to the topology for \( \Omega_0 \). Since \( \Omega_0 \) is a regular Hausdorff space, the collection \( N(\sigma) \) of closed neighborhoods of \( \sigma \) is a base for the neighborhood system at \( \sigma \). Now if the singleton \( \{ \sigma \} \) is open, then there exists \( k \in D \) such that \( \sigma_n = \sigma \) for all \( n > k \); hence, the net \( \sigma \)-converges to \( \sigma \). On the other hand, if the singleton \( \{ \sigma \} \) is not open, then (putting \( \sigma = (0, x) \)) we have \( \inf \{ (+1, E) : E \in N(x) \} = \sup \{ (-1, E) : E \in N(x) \} = (0, x) = \sigma \), where \( N(x) \) is the collection of closed neighborhoods of \( x \in X \). Hence, for each \( E \in N(x) \) there exists \( k \in D \) such that \( (-1, E) \subseteq \sigma_n \subseteq (+1, E) \) for all \( n > k \). Therefore, the net \( \sigma \)-converges to \( \sigma \).

Now let \( \{ \sigma_n : n \in D \} \) be a net in \( \Omega_0 \) which does not converge to \( \sigma \) with respect to the topology for \( \Omega_0 \). Hence, there must exist a set \( E \in \mathcal{F} \) which does not contain \( x \), where \( (0, x) = \sigma \), such that \( x_n \in E \) co-finually \( (\sigma_n = (0, x_n)) \). Hence, \( \tau_k = \inf \{ \sigma_n : n > k \} \leq (+1, E) \) for all \( k \in D \). Therefore, \( \sup \{ \tau_k : k \in D \} \leq (+1, E) \) which means that \( \sup \{ \tau_k : k \in D \} \neq \sigma \) (recall the definition of the partial order in \( \Omega \) and the fact that \( x \) does not belong to \( E \)). This in turn means that the net does not \( \sigma \)-converge to \( \sigma \). Q.E.D.

References


University of Washington