AN EXACT SEQUENCE IN GALOIS COHOMOLOGY

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Let $A$ be an integrally closed noetherian domain with the quotient field $F$. The group of divisors of $A$ is the free abelian group generated by nonzero minimal prime ideals of $A$ and is denoted by $D(A)$. This is canonically isomorphic to the group gotten from the set of all reflexive $A$-ideals (including fractional ideals) under the rule $a \cdot b = (a \cdot b)^*$ where $^* = \{a \in F \mid ac \subset A\} = \text{Hom}_A(c, A)$. The divisor class group of $A$ denoted by $C(A)$ is the factor group of $D(A)$ by the principal divisors, i.e. it is defined by the exact sequence

$$0 \rightarrow U(F)/U(A) \rightarrow D(A) \rightarrow C(A) \rightarrow 0$$

where $D(a) = \sum_p p_a(a)p$ with $(pA^*)^a_a = aA^*$. We observe that $A$ is a unique factorization domain if and only if $C(A) = 0$, i.e. if and only if $U(F)/U(A) \rightarrow D(A)$ is an isomorphism.

Now let $S \supset R$ be an integral extension of an integrally closed noetherian domain, whose quotient field $L \supset K$ is a separable extension of finite degree. Then we obtain the canonical map $i: D(R) \rightarrow D(S)$ given by $\sum p \rightarrow \sum p e(p(\mathfrak{F})/\mathfrak{F})$ where $e(\mathfrak{F})$ is the ramification index of $\mathfrak{F}$ in $S \supset R$, i.e. $pS\mathfrak{F} = (\mathfrak{F}S\mathfrak{F})^{e(\mathfrak{F})}$. Since the map $i$ sends the principal divisors to principal divisors, it induces the map $i: C(R) \rightarrow C(S)$. We denote the kernel of $i$ by $C(S/R)$. Thus $C(S/R)$ is the subgroup of $C(R)$ consisting of those divisor classes which become principal under the extension $S \supset R$. Now let the quotient field extension $L \supset K$ be Galois with the Galois group $G$. As customary we denote $H^n(G, U(L))$, $H^n(G, U(S))$ by $H^n(L/K)$, $H^n(S/R)$ respectively. The main purpose of this short note is to prove:

**Theorem.** Let $S \supset R$ be an integral extension of an integrally closed noetherian domain whose quotient field extension $L \supset K$ is Galois with the Galois group $G$. Then we have the exact sequence

$$0 \rightarrow C(S/R) \rightarrow H^1(S/R) \rightarrow D(S)/iD(R) \rightarrow C(S)/iC(R) \rightarrow$$

$$\rightarrow H^1(S/R) \rightarrow \bigcap D(S_{R}/R) \rightarrow H^1(G, C(S)) \rightarrow H^1(S/R)$$

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2 For any commutative ring $A$, we denote by $U(A)$ the group of invertible elements in $A$. 

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where \( \cap_{\mathfrak{p}} H^2(S_{\mathfrak{p}}/R_{\mathfrak{p}}) = \cap_{\mathfrak{p}} \text{Im}(H^3(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^3(L/K)) \), \( \mathfrak{p} \) running through all nonzero minimal primes of \( R \).

**Remark.** A somewhat similar exact sequence related to the Brauer groups in the case when \( S \supset R \) is unramified was obtained in [2], [3].

**Proof.** Firstly we observe that \( H^1(G, D(S)) = 0 \). Indeed, if we fix, for each nonzero minimal prime \( \mathfrak{p} \) in \( R \), a nonzero minimal prime \( \mathfrak{q} \) in \( S \) lying above \( \mathfrak{p} \), and if we denote by \( G_{\mathfrak{p}} \) the decomposition subgroup of \( \mathfrak{q} \) over \( \mathfrak{p} \), then \( D(S) \cong \sum_{\mathfrak{p}} \mathbb{Z}[G_{\mathfrak{p}}] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \) as \( G \)-modules, where \( \mathfrak{p} \) runs through all nonzero minimal primes of \( R \). Consequently \( H^*(G, D(S)) = \sum_{\mathfrak{p}} H^*(G_{\mathfrak{p}}, \mathbb{Z}) \) and in particular we have \( H^1(G, D(S)) = 0 \). Now for each minimal prime \( \mathfrak{p} \) in \( R \), \( S_{\mathfrak{p}} \) is a unique factorization domain and hence \( 0 \to U(S_{\mathfrak{p}}) \to U(L) \to D(S_{\mathfrak{p}}) \to 0 \) is an exact sequence of \( G \)-modules. Therefore \( 0 \to H^3(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^3(L/K) \to H^3(G, D(S_{\mathfrak{p}})) \) is exact and hence we obtain the exact sequence

\[
0 \to \bigcap_{\mathfrak{p}} H^3(S_{\mathfrak{p}}/R_{\mathfrak{p}}) \to H^3(L/K) \to H^3(G, D(S)).
\]

The exact sequence of \( G \)-modules \( 0 \to U(L)/U(S) \to D(S) \to C(S) \to 0 \) together with \( H^1(G, D(S)) = 0 \) gives us the exact sequences

\[
0 \to (U(L)/U(S))^a \to D(S)^a \to C(S)^a \to H^1(G, U(L)/U(S)) \to 0,
\]

\[
0 \to H^1(G, C(S)) \to H^1(G, U(L)/U(S)) \to H^2(G, D(S)).
\]

In turn the exact commutative diagram

\[
0 \to U(K)/U(R) \to D(R) \to C(R) \to 0
\]

\[
0 \to (U(L)/U(S))^a \to D(S)^a \to C(S)^a \to H^1(G, U(L)/U(S)) \to 0
\]

yields the exact sequence

\[
0 \to C(S/R) \to (U(L)/U(S))^a/(U(K)/U(R)) \to D(S)^a/iD(R)
\]

\[
\to C(S)^a/iC(R) \to H^1(G, U(L)/U(S)) \to 0.
\]

On the other hand, the exact sequence \( 0 \to U(S) \to U(L) \to U(L)/U(S) \to 0 \) together with Hilbert's Theorem 90 gives us the exact sequences

\[
0 \to U(R) \to U(K) \to (U(L)/D(S))^a \to H^1(S/R) \to 0,
\]

\[
0 \to H^1(G, U(L)/U(S)) \to H^3(S/R) \to H^3(L/K)
\]

\[
\to H^2(G, U(L)/U(S)) \to H^3(S/R) \to \cdots.
\]

Now the exact commutative diagram
0 \to H^1(G, U(L)/U(S)) \to
\downarrow
H^1(S/R) \to H^1(L/K) \to H^1(G, U(L)/U(S)) \to H^2(S/R)
\downarrow
0 \to H^2(G, D(S)) \to H^3(S/R)
\text{together with (1) and (3) yields the exact sequence}
0 \to H^1(G, U(L)/U(S)) \to H^2(S/R) \to \bigcap_b H^2(S_b/R_b)
(7)
\to H^1(G, C(S)) \to H^2(S/R).

On the other hand, (4) and (5) gives us
0 \to C(S/R) \to H^1(S/R) \to D(S)/iD(R) \to C(S)/iC(R)
(8)
\to H^1(G, U(L)/U(S)) \to 0.

Connecting (7) and (8) we obtain the desired exact sequence.

When $S \supset R$ is unramified, we can relate the 2-dimensional cohomology group with the Brauer group. We denote by $B(S/R)$ the kernel of the canonical map $B(R) \to B(S)$, where $B( )$ denotes the Brauer group. Then $H^2(S/R) = B(S/R)$ if $S \supset R$ is unramified and $R$ is a local domain [1] and thus we obtain:

**Corollary 1.** Let $S \supset R$ be unramified. Then we have the exact sequence

$$0 \to H^1(S/R) \to C(R) \to C(S)^g \to H^2(S/R) \to \bigcap_b B(S_b/R_b)$$
$$\to H^1(G, C(S)) \to H^2(S/R).$$

**Proof.** If $S \supset R$ is unramified, then $D(S)^g = iD(R)$ and $\bigcap_b H^2(S_b/R_b) = \bigcap_b B(S_b/R_b)$.

If we further assume $R$ to be regular, our exact sequence coincides with the exact sequence in [2], [3].

**Corollary 2.** Let $S \supset R$ be unramified. If $R$ is regular, we have the exact sequence

$$0 \to H^1(S/R) \to C(R) \to C(S)^g \to H^2(S/R) \to B(S/R)$$
$$\to H^1(G, C(S)) \to H^2(S/R).$$

**Proof.** We must show that $\bigcap_b B(S_b/R_b) = B(S/R)$. Since one side inclusion is clear [1], it suffices to show that $\bigcap_b B(S_b/R_b) \subset B(S/R)$, i.e. $\text{Ker}(H^2(L/K) \to H^2(G, D(S)) \subset B(S/R)$. Now $S$, being unramified over a regular domain $R$, is also regular and hence is a local unique factorization domain. Consequently $D(S)$ is nothing but the group of
invertible $S$-ideals. Let $\alpha \in \text{Ker}(H^2(L/K) \to H^2(G, D(S)))$, and let \( \{a_{x,r}\} \) be a 2-cocycle representing $\alpha$. This means that there exists a set \( \{A_x\} \) of invertible $S$-ideals indexed by $G$ such that $a_{x,r} A_x A_x^{-1} A_{x^{-1}} = S$, i.e. $a_{x,r} A_x A_x^{-1} = A_{x^{-1}}$. (We may assume that $A_1 = S$.) Now let $\Gamma$ be the central simple $K$-algebra associated with the 2-cocycle \( \{a_{x,r}\} \), i.e. $\Gamma = \sum \langle L \rangle$, (direct sum) with the multiplication rule: $(x, u_e)(x, u_e) = x_x u_x a_{x, u_e}$. Then $\Lambda = \sum_x A_x$ which is a subset of $\Gamma$ is stable under the multiplication since $A_x u_e A_x u_e = A_x A_x^{-1} A_{x^{-1}} u_e = A_{x^{-1}} u_e$. Thus $\Lambda$ is an order over $R$ in $\Gamma$, and is projective as an $R$-module. Now consider the canonical map $S \otimes_R \Lambda \to \text{Hom}_R(\Lambda, \Lambda)$ given by $(s \otimes \lambda)(x) = sx \lambda$. Since $\alpha \in \text{Ker}(H^2(L/K) \to H^2(G, D(S))) = \cap_p H^2(S_p/R_p)$, it follows that $S_p \otimes \Lambda_p \to \text{Hom}_{S_p}(\Lambda_p, \Lambda_p)$ is an isomorphism for all non-zero minimal primes $p$ in $R$. Consequently the canonical map $S \otimes_R \Lambda \to \text{Hom}_R(\Lambda, \Lambda)$ is an isomorphism since both sides are $R$-projective modules of the same rank. Therefore $\Lambda$ is an $R$-separable order in $\Sigma$ with $S$ as a splitting ring, and this completes our proof.

Bibliography


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