A REMARK ON THE GENERAL
SUMMABILITY THEOREM¹

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1. Introduction. The following theorem is known as the general
summability theorem in the theory of Fourier integrals (see Titch-
marsh [4, p. 28]).

Theorem A. Suppose that a function $K(x, y, \delta)$ of $y$ belongs to
$L^1(-\infty, \infty)$ and satisfies, for a fixed $x$,
\begin{align}
(1.1) & \quad K(x, y, \delta) = O(1/\delta), \quad \text{for} \quad |y - x| \leq \delta,
(1.2) & \quad = O(\delta^\alpha/|x - y|^{|1+\alpha|}), \quad \text{for} \quad |x - y| > \delta,
\end{align}
for some positive $\alpha$ and
\begin{align}
(1.3) & \quad \lim_{\delta \to 0} \int_x^\infty K(x, y, \delta) \, dy = 1/2,
(1.4) & \quad \lim_{\delta \to 0} \int_{-\infty}^{-x} K(x, y, \delta) \, dy = 1/2.
\end{align}

Let $f(y)/(1 + |y|^{\alpha+1})$ belong to $L^1(-\infty, \infty)$. Then
\begin{align}
(1.5) & \quad \lim_{\delta \to 0} \int_{-\infty}^{\infty} K(x, y, \delta)f(y) \, dy = (1/2)\{\phi(x) + \psi(x)\},
\end{align}
wherever
\begin{align}
(1.6) & \quad \int_0^h \left| f(x + t) - \phi(x) \right| \, dt = O(h)
\end{align}
and
\begin{align}
(1.7) & \quad \int_0^h \left| f(x - t) - \psi(x) \right| \, dt = O(h)
\end{align}
as $h \to 0+$. This implies Fejér's integral theorem and other kinds of ordinary
summability theorems. However, Theorem A is not true if $\alpha=0$ and
as a matter of fact, it does not imply the Fourier single integral theo-
rem. In this paper we shall prove a theorem which corresponds to the

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case when \( \alpha = 0 \) and implies as a particular case the ordinary Fourier single integral theorem.

2. Theorem and its proof. We shall prove the following

**Theorem.** Suppose that for a fixed \( x \) the improper integral \( \int_{-\infty}^{\infty} K(x, y, \delta) \, dy \) exists. Let for fixed \( x \) and \( \eta > 0 \),

\[
(2.1) \quad K(x, y, \delta) = O(|x - y|^{-1}), \quad \text{for} \quad |x - y| > \eta > 0,
\]

where \( O \) is independent of \( \delta \) (but may depend on \( \eta \)).

Suppose further that

\[
(2.2) \quad \lim_{\delta \to 0} \int_{-\infty}^{\infty} K(x, y, \delta) \, dy = \rho,
\]

\[
(2.3) \quad \lim_{\delta \to 0} \int_{-\infty}^{\infty} K(x, y, \delta) \, dy = 1 - \rho,
\]

where \( 0 < \rho < 1 \);

\[
(2.4) \quad \lim_{\delta \to 0} \int_{\beta}^{\infty} K(x, y, \delta) \, dy = 0, \quad \text{for every constant} \ \beta > x,
\]

\[
(2.5) \quad \lim_{\delta \to 0} \int_{-\infty}^{\alpha} K(x, y, \delta) \, dy = 0, \quad \text{for every constant} \ \alpha < x;
\]

\[
(2.6) \quad \lim_{\delta \to 0} \int_{\alpha}^{\beta} (y - x) K(x, y, \delta) \, dy = 0,
\]

for every pair \( \alpha, \beta \) of constants such that \( x < \alpha < \beta \) or \( \alpha < \beta < x \); and

\[
(2.7) \quad \int_{\alpha}^{\beta} K(x, y, \delta) \, dy = O(1)
\]

for every constant \( \beta \), \( O \) being independent of \( \delta \).

Let \( f(y)/(1 + |y|) \in L^1(-\infty, \infty) \) and let \( f(y) \) be of bounded variation in a neighborhood of \( x \). Then we have

\[
(2.8) \quad \lim_{\delta \to 0} \int_{-\infty}^{\infty} f(y) K(x, y, \delta) \, dy = \rho f(x + 0) + (1 - \rho)f(x - 0).
\]

Note that the integral on the left-hand side is absolutely convergent, because using (2.1) we have

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* I missed the conditions (2.4) and (2.5) in the original form of the theorem. I am obliged to Professor S. Bochner for calling this point to my attention.
\[ \int_A^\infty |f(y)K(x, y, \delta)| \, dy \leq \int_A^\infty |f(y)| \frac{1}{|x - y|} O(1) \, dy \]

for a large \( A \). The same thing is true for \( \int_{-A}^0 \).

**Proof of the theorem.** It is sufficient to prove that

\[ \lim_{\delta \to 0} \left\{ \int_0^\infty K(x, y, \delta)f(y) \, dy - f(x + 0) \int_0^\infty K(x, y, \delta) \, dy \right\} = 0, \]

together with a similar result in terms of \( f^- \) and \( f^+ \).

We may suppose without loss of generality that \( f(y) \) is nondecreasing in a right-hand neighborhood of \( x \). For any given positive \( \epsilon \) we choose \( \eta \) such that

\[ |f(y) - f(x + 0)| < \epsilon, \quad \text{for } x \leq y \leq x + \eta, \]

and we write the expression in (2.9) in the following way.

\[
I = \int_{x+\eta}^{x+\eta} K(x, y, \delta) \{f(y) - f(x + 0)\} \, dy \\
+ \int_{x+\eta}^{x+\eta} K(x, y, \delta)f(y) \, dy - f(x + 0) \int_{x+\eta}^{x+\eta} K(x, y, \delta) \, dy \\
= I_1 + I_2 + I_3, 
\]
say.

For every fixed \( \eta > 0 \), we have, because of (2.4),

\[ \lim_{\delta \to 0} I_3 = 0. \]

The second mean value theorem shows that

\[ |I_1| \leq \epsilon \int_{x+\xi}^{x+\eta} K(x, y, \delta) \, dy \]

for some \( 0 < \xi < \eta \). Hence by (2.7), there exists a constant \( C \) independent of \( \delta \) such that

\[ |I_1| \leq C\epsilon. \]

In order to handle \( I_2 \), we split it into two parts

\[
I_2 = \int_{x+\eta}^{x+A} K(x, y, \delta)f(y) \, dy + \int_{x+\eta}^{x+A} K(x, y, \delta)f(y) \, dy \\
= I_{21} + I_{22}, \quad \eta < A.
\]

Now
where $C_1$ is a constant and the integral on the right-hand side may be made as small as desired by taking $A$ large, since $f(\gamma)/(1 + |\gamma|) \in L^1(-\infty, \infty)$. Hence we may write

$$|I_{22}| \leq C_2 \varepsilon,$$

where $C_2$ is a constant.

Now we choose a step-function $g(\gamma)$ in $(x + \eta, x + A)$, with a finite number of jumps, in such a way that

$$\int_{x+\eta}^{x+A} \left| \frac{f(\gamma)}{y - x} - g(\gamma) \right| d\gamma < \varepsilon$$

for fixed $A$ and $\eta$. We then write

$$I_{21} = \int_{x+\eta}^{x+A} (y - x)K(x, \gamma, \delta) \left( \frac{f(\gamma)}{y - x} - g(\gamma) \right) d\gamma$$

$$+ \int_{x+\eta}^{x+A} (y - x)K(x, \gamma, \delta)g(\gamma) d\gamma.$$

Then

$$|I_{21}| \leq C \int_{x+\eta}^{x+A} \left| \frac{f(\gamma)}{y - x} - g(\gamma) \right| d\gamma + \int_{x+\eta}^{x+A} g(\gamma)(y - x)K(x, \gamma, \delta) d\gamma$$

so that

$$|I_{21}| \leq C \varepsilon + \int_{x+\eta}^{x+A} g(\gamma)(y - x)K(x, \gamma, \delta) d\gamma.$$

We shall show that the last integral converges to zero as $\delta \to 0$. To do this it is sufficient to prove that

$$\int_{x+\eta}^{x+A} g(\gamma)(y - x)K(x, \gamma, \delta) d\gamma \to 0, \text{ for } x + y \leq \alpha < \beta \leq x + A$$

which is no more than the condition (2.6).

Combining (2.11), (2.12), (2.13), (2.14) and (2.15) we have

$$\lim_{\delta \to 0} \sup_{x \to 0} |I| \leq (2C + C_2) \varepsilon.$$

The same thing is true for (2.9) with $(-\infty, x)$ and $f(x - 0)$. This proves the theorem.
3. Remarks. If $K(x, y, \delta) = \lambda K(\lambda(y-x))$ with $\delta = 1/\lambda$ and $K(x)$ is an even function, then (2.8) becomes

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} f\left(x + \frac{u}{\lambda}\right) K(u) \, du = \frac{1}{2} \left\{ f(x + 0) + f(x - 0) \right\}.$$  

(3.1)

The theorem states that (3.1) is true if $f(u)/(1 + |u|) \in L^1(-\infty, \infty)$, $f(u)$ is a function of bounded variation in a neighborhood of $x$ and $K(u)$ satisfies the conditions that

$$\int_{-\infty}^{\infty} K(u) \, du = 1,$$

(3.2)

$$\lambda \int_{\beta}^{\infty} K(\lambda u) \, du \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \text{for every} \beta > 0,$$

(3.3)

$$\lim_{\lambda \to \infty} \lambda \int_{\alpha}^{\beta} uK(\lambda u) \, du = 0, \quad \text{if} \quad \beta > \alpha > 0 \quad \text{or} \quad \alpha < \beta < 0,$$

(3.4)

$$\int_{0}^{\beta} uK(u) \, du = O(1), \quad \text{for every} \beta,$$

(3.5)

$$uK(u) = O(1).$$

(3.6)

Since the function $\sin u/(\pi u)$ satisfies above conditions, (3.1) gives us the Fourier single integral theorem. As to the above form (3.1) of summability theorem, one may refer to Bochner [1], Bochner-Chandrasekharan [2] and Bochner-Izumi [3]. It will be noted that the proof given in this paper is just a generalization of the proof of the Fourier single integral theorem. $I_1$ and $I_2$ have been handled in the usual manner. In proving that $I_2$ converges to zero with the Dirichlet integral, the Riemann-Lebesgue lemma is usually used, but the method employed here is known to be applicable in proving the Riemann-Lebesgue lemma as observed by Bochner and Chandrasekharan [1, p. 4].

REFERENCES


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