STONE-WEIERSTRASS THEOREMS FOR THE STRICT TOPOLOGY

CHRISTOPHER TODD

1. Let $X$ be a locally compact Hausdorff space, $E$ a (real) locally convex, complete, linear topological space, and $(C^*(X, E), \beta)$ the locally convex linear space of all bounded continuous functions on $X$ to $E$ topologized with the strict topology $\beta$. When $E$ is the real numbers we denote $C^*(X, E)$ by $C^*(X)$ as usual. When $E$ is not the real numbers, $C^*(X, E)$ is not in general an algebra, but it is a module under multiplication by functions in $C^*(X)$.

This paper considers a Stone-Weierstrass theorem for $(C^*(X), \beta)$, a generalization of the Stone-Weierstrass theorem for $(C^*(X, E), \beta)$, and some of the immediate consequences of these theorems. In the second case (when $E$ is arbitrary) we replace the question of when a subalgebra generated by a subset $S$ of $C^*(X)$ is strictly dense in $C^*(X)$ by the corresponding question for a submodule generated by a subset $S$ of the $C^*(X)$-module $C^*(X, E)$. In what follows the symbols $C_0(X, E)$ and $C_{00}(X, E)$ will denote the subspaces of $C^*(X, E)$ consisting respectively of the set of all functions on $X$ to $E$ which vanish at infinity, and the set of all functions on $X$ to $E$ with compact support.

Recall that the strict topology is defined as follows [2]:

**Definition.** The strict topology $(\beta)$ is the locally convex topology defined on $C^*(X, E)$ by the seminorms

$$
\|f\|_{*v} = \sup_{x \in X} |\phi(x)f(x)|,
$$

where $v$ ranges over the indexed topology on $E$ and $\phi$ ranges over $C_0(X)$.

When $E$ is the real numbers the above definition may be given simply as: a net $\{f_\alpha: \alpha \in A\}$ converges strictly to a function $f$ iff $\{\phi f_\alpha: \alpha \in A\}$ converges uniformly to $\phi f$ for each $\phi$ in $C_0(X)$. It is known that $C^*(X, E)$ is complete in the strict topology [2].

2. **Strict approximation in $C^*(X)$.** Stone-Weierstrass theorems were given for $C^*(X)$, under special restrictions by Buck [2]. The exact analogue of the classical theorem (for complex valued functions) was obtained by Glicksberg [3] as a corollary to a version of Bishop's generalized Stone-Weierstrass theorem [1]. The proof is not elementary. We give here a proof which follows quite simply from Buck's

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original theorems. These theorems were proved under the additional hypothesis that either (a) the subalgebra in question contained a function vanishing nowhere or (b) the space X on which the functions were defined was σ-compact.

**Theorem 1.** Let X be a locally compact Hausdorff space and let \( \mathfrak{A} \) be a strictly closed subalgebra of \( C^*(X) \) which separates points of X and which for each \( x \) in X, contains a function \( g \) with \( g(x) \neq 0 \), then \( \mathfrak{A} = C^*(X) \).

**Proof.** Let \( f \in C^*(X) \) and let \( \phi \in C_0(X) \) and \( \epsilon > 0 \) be given. Let \( K(\phi) \) be the σ-compact set outside of which \( \phi \) vanishes identically. \( K(\phi) \) may be replaced by a regularly σ-compact subset \( K \) of X (i.e. \( K \) is a countable union of compact subsets \( K_n \), with \( K_n \) contained in the interior of \( K_{n+1} \)) which contains \( K(\phi) \) and which is an open \( F_\sigma \) set [2]. \( K \) is a locally compact Hausdorff space in its relative topology and hence the strict topology is defined for \( C^*(K) \), and \( \phi \) is in \( C_0(K) \). \( \mathfrak{A} \) restricted to \( K \) is dense in \( C^*(K) \) in the strict topology by (b) above, consequently there is a function \( g \) in \( \mathfrak{A} \) such that \( \| \phi(x)f(x) - \phi(x)g(x) \| < \epsilon \) for \( x \) in \( K \). Since \( \phi f \) and \( \phi g \) vanish identically outside \( K \), we have \( \| g - f \|_\epsilon < \epsilon \). The conclusion now follows from the fact that \( (C^*(X), \beta) \) is complete.

The usual theorem for the complex case follows immediately from the above. We also note without proof the following obvious consequences of Theorem 1. (1) If \( X \) is a product of locally compact Hausdorff spaces which is locally compact, then every function in \( C^*(X) \) may be strictly approximated by finite sums of finite products of functions of one variable on X. (2) There is a 1-1 correspondence between the β-closed ideals of \( C^*(X) \) and the closed subsets of X. In particular there is a 1-1 correspondence between the maximal β-closed ideals and the points of X, and each β-closed ideal is the intersection of all β-closed maximal ideals which contain it.

Theorem 1 may be applied to obtain the following result.

**Theorem 2.** If \( X \) is a locally compact and σ-compact Hausdorff space, then \( (C^*(X), \beta) \) is separable iff \( X \) is metrizable.

**Proof.** Suppose that \( X \) is metrizable, hence since \( X \) is σ-compact, \( X \) is separable and satisfies the second axiom of countability. Let \( \mathfrak{B} \) be a countable base for the topology on X. For each \( B \) in \( \mathfrak{B} \), the complement of \( B \), being closed, is the zero set of a function \( f_B \) in \( C^*(X) \). Let \( \mathfrak{A} \) be the countable algebra generated by \( \{ f_B | B \in \mathfrak{B} \} \) over the rationals. \( \mathfrak{A} \) clearly satisfies the conditions of Theorem 1 and hence is β-dense in \( C^*(X) \). Conversely let \( S \) be a countable β-dense subset
of $C^*(X)$, and let $\mathcal{A}_S$ be the countable algebra generated by $S$ over the rationals. Let $\phi$ be any element of $C_0(X)$ which vanishes nowhere on $X$. Let $\phi \mathcal{A}_S = \{ \phi f | f \in \mathcal{A}_S \} \subseteq C_0(X)$. By the classical Stone-Weierstrass theorem $\phi \mathcal{A}_S$ is uniformly dense in $C_0(X)$. Let $\mathcal{A}^*_S$ be the subset of $\phi \mathcal{A}_S$ consisting of all functions $f$ such that $\|f\| \leq 1$. These functions may be used to define a function $E$ on $X$ into the unit cube $Q^2 = \prod \{ [0, 1] | f \in \mathcal{A}^*_S \}$ by $E(x)_f = f(x)$. $E$ is easily shown to be a homeomorphism, hence $X$ is metrizable.

It is conjectured that separability for $(C^*(X), \beta)$ under the sole hypothesis that $X$ is locally compact and Hausdorff is characterized by the statement: $(C^*(X), \beta)$ is separable iff $X$ is metrizable and separable. Note that if $X$ is metrizable and separable then $X$ is necessarily $\sigma$-compact.

3. **Strict approximation in $C^*(X, E)$**. Let $S$ be a subset of $C^*(X, E)$ and let $x$ be an element of $X$, $M$ a closed subspace of $E$. If $S(x) = \{ \alpha \in E | \alpha = f(x) \text{ for some } f \in S \}$ is contained in $M$ then we will say $S$ is restricted to $M$ at $x$. We will call a subset $S$ of $C^*(X, E)$ unrestricted if for each maximal closed subspace $M$ of $E$ and each point $x$ of $X$, there is a function $f$ in $S$ with $f(x) \in M$.

**Theorem 3.** Let $X$ be a locally compact Hausdorff space, and $E$ a locally convex, complete, linear topological space. If $S$ is an unrestricted submodule of $C^*(X, E)$, then $S$ is strictly dense in $C^*(X, E)$.

We first give a proof of the following fact due to Buck [2].

**Lemma.** $C_{00}(X, E)$ is strictly dense in $C^*(X, E)$.

**Proof.** Let $f \in C^*(X, E)$, $\phi \in C_0(X)$, a seminorm $\nu$ for $E$ and $\epsilon > 0$ be given. $\phi f$ is in $C_0(X, E)$. Let $K \subseteq X$ be the compact set outside of which $|\phi f(x)| \leq \epsilon/2$. There exists a function $\theta$ in $C_0(X)$ such that $\theta(x) = 1$ for $x$ in $K$ and $\|\theta\| \leq 1$. $\theta f$ is in $C_0(X, E)$ and $|\phi \theta f(x) - \phi f(x)| \leq \epsilon$ for every $x$ in $X$. Therefore $f$ is in the strict closure of $S$.

**Proof of Theorem 3.** In view of the preceding lemma we need only show that if $S$ is an unrestricted submodule, then $C_{00}(X, E)$ lies within the strict closure of $S$. We show in fact that $C_{00}(X, E)$ is in the uniform closure of $S$ (it is true that $C_0(X, E)$ is in the uniform closure of $S$).

Let $f$ be any function in $C_0(X, E)$ and let a seminorm $\rho$ for $E$ and $\epsilon > 0$ be given. Let $\Psi_\rho(x) = |f(x)|_\rho$. $\Psi_\rho$ is in $C_0(X)$. Let $K$ be the compact set outside of which $\Psi_\rho$ vanishes identically. For each $g$ in $S$, $\Psi_\rho g$ is in $S$ and $\Psi_\rho g(x) = 0$ for $x$ not in $K$. Let $x_0$ be any point of $K$ and suppose $f(x_0) = \xi$. By hypothesis there is a function $g$ in $S$ such that $g(x_0) = \xi$. Let $\Psi_\rho(x_0) = a$. If $a \neq 0$ let $h_{x_0}$ be the function $a^{-1}\Psi_\rho g$. $h_{x_0}(x_0)$
= \xi, \ h_{x_0}(x) = 0 \text{ for } x \notin K, \text{ and } h_{x_0} \in S. \text{ Thus there exists a neighborhood } U_{x_0} \text{ of } x_0 \text{ such that } |h_{x_0}(y) - f(y)|_\rho < \epsilon \text{ for } y \in U_{x_0}. \text{ Suppose that } \Psi_x(x_0) = 0. \text{ Then let } h_{x_0} = \Psi_xg. \ |h_{x_0}(x_0) - f(x_0)|_\rho = |f(x_0)|_\rho = 0 \text{ and again by the continuity of seminorms we may choose a neighborhood } U_{x_0} \text{ of } x_0 \text{ with } |h_{x_0}(y) - f(y)|_\rho < \epsilon \text{ for } y \in U_{x_0}. \text{ In the usual manner we obtain a finite number of points } x_i \text{ in } K, \text{ of functions } h_i \in S, \text{ and neighborhoods } U_i \text{ of } x_i \text{ such that if } y \text{ is in } K \text{ then } y \text{ is in } U_i \text{ for some } i \text{ and } |h_y(y) - f(y)|_\rho < \epsilon. \text{ The set of neighborhoods } U_i \text{ is a point finite open cover of } K \text{ in its relative topology. Since } K \text{ is completely regular and compact, there exist functions } \phi_i \in C^*(K) = C(K) \text{ such that } \phi_i(U_i) \subseteq [0, 1], \phi_i \text{ vanishes identically in } K \text{ outside } U_i \cap K, \text{ and } \sum_i \phi_i(x) = 1 \text{ for each } x \in K. \text{ Since } K \text{ is a compact subset of a completely regular space each } \phi_i \text{ may be extended to a function in } C^*(X), \text{ so that the functions } \phi_i h_i \text{ are defined for each } i \text{ and are in } S. \text{ The function } F = \sum_i \phi_i h_i \text{ is in } S, \text{ is identically 0 outside } K \text{ and } \|F - f\|_\rho < \epsilon. \text{ Thus } C_{00}(X, E) \text{ lies in the uniform closure of } S. \text{ From the preceding lemma it follows that } S \text{ is strictly dense in } C^*(X, E).

**Corollary 1.** Any subspace of } C^*(X, E) \text{ which is a } C_{00}(X) \text{ submodule is strictly dense in } C^*(X, E). \text{ A strictly closed } C_{00}(X) \text{ submodule is a } C^*(X) \text{ module.}

**Corollary 2.** A function } f \in C^*(X, E) \text{ is in the strict closure of the } C^*(X) \text{ submodule generated by an arbitrary subset } S \text{ of } C^*(X, E) \text{ if and only if for any } x \in X \text{ and maximal closed subspace } M \text{ of } E, S(x) \subseteq M \text{ implies } f(x) \in M.

Theorem 3 generalizes results obtained in [2]. We use the above results to characterize the strictly closed maximal submodules of } C^*(X, E). \text{ If } M \text{ is a maximal closed subspace of } E \text{ and } x \text{ is a fixed point of } X, \text{ let } S_{x,M} = \{f \in C^*(X, E) | f(x) \in M\}. \text{ Let } y \text{ be any point of } X \text{ with } y \neq x, \text{ and } a \in E, b \in M. \text{ Let } \phi_1 \text{ and } \phi_2 \text{ be functions in } C^*(X) \text{ such that } \phi_1(y) = 1, \phi_1(x) = 0, \text{ and } \phi_2(y) = 0, \phi_2(x) = 1. \text{ Let } h = a\phi_1 + b\phi_2, \ h(x) = b, \ h(y) = a \text{ and } h \text{ is in } S_{x,M}. \text{ Thus } S_{x,M}(x) = M \text{ and } S_{x,M}(y) = E \text{ for each } y \text{ in } X \text{ with } y \neq x. \text{ By Corollary 2, } S_{x,M} \text{ is a strictly closed submodule of } C^*(X, E). \text{ We show that it is a maximal submodule. Suppose that } S' \text{ is a submodule of } C^*(X, E) \text{ such that } S' \supset S_{x,M}, S' \neq S_{x,M}. \text{ Then there is a function } f \in S' \text{ such that } f(x) \notin M. \text{ Let } f(x) = \lambda \in E. \text{ E may be represented as a direct sum } M \oplus N \text{ where } N \text{ is a complement of } M. \text{ Then for each } \xi \in E, \xi = m + n, \text{ with } m \in M, n \in N, \text{ and so for some } m_0 \text{ in } M \text{ and } n_0 \text{ in } N, \lambda = m_0 + n_0. \text{ There exists a function } g \text{ in } S_{x,M} \text{ such that } g(x) = m_0. \text{ The function } f' = f - g \text{ in } S' \text{ has the property } f'(x) = n_0.
Since $M$ is of deficiency 1 in $E$ each point of $N$ is of the form $\alpha n_0$, where $\alpha$ is a real number. Thus if $\xi$ is any element of $E$, $\xi = m + \alpha n_0$, there is a function $h$ in $S'$, $h = g + \alpha f'$ (where $g$ is a function in $S_{x,M}$) such that $h(x) = \xi$, hence $S'(x) = E$. Also $S'(y) = E$ for any point $y$ in $X$. Now let $j$ be any function in $C^*(X, E)$ and let $j(x) = \eta$. There is a function $g_1$ in $S'$ with $g_1(x) = \eta$. Let $g_2 = j - g_1$. $g_2(x) = 0$ so $g_2$ is in $S_{x,M} \subseteq S'$. $j = g_1 + g_2$ so $j$ is in $S'$ thus $S'$ is $C^*(X, E)$. Consequently $S_{x,M}$ is a strictly closed maximal submodule. Conversely it is easy to see that if $S$ is a strictly closed maximal submodule of $C^*(X, E)$ then $S$ is of the form $S_{x,M}$.

Note that unlike the situation for maximal ideals there may exist maximal submodules which are not strictly closed. Let $E$ be any space (with the usual requirements) which admits a discontinuous linear functional $L$. Let $N$ be the null space of $L$. $N$ is a maximal subspace of $E$ and is dense in $E$. Let $x$ be any point of $X$ and let $S$ be the set of all functions $f$ in $C^*(X, E)$ such that $f(x) \in N$. It is a routine verification that $S$ is a $C^*(X)$ submodule of $C^*(X, E)$. We show that $S$ is a maximal submodule. Let $\eta$ be any element of $E$ with $\eta$ not in $N$ and let $f$ be a function in $C^*(X, E)$ with $f(x) = \eta$. If $\xi$ is any point of $E$ then $\xi = \alpha \eta + n$ with $n \in N$ since $N$ is maximal. Let $g$ be any function in $C^*(X, E)$ and let $g(x) = \lambda = \partial \eta + n$, $n \in N$. Let $g_1 = g - \partial f$. $g_1(x) = g(x) - \partial f(x) = n$ so $g_1$ is in $S$. $g = g_1 + \partial f$ so $S$ is a maximal submodule. $S$ is not strictly closed, however, by Corollary 2 of Theorem 3. We state the preceding facts as

**Theorem 4.** Let $X$ be a locally compact Hausdorff space, $E$ a locally convex, complete linear space. The strictly closed maximal submodules of $C^*(X, E)$ are the submodules of the form $S_{x,M}$. If $E$ admits no discontinuous linear functional then every maximal submodule is closed.

**Theorem 5.** Let $X$ be a locally compact Hausdorff space and $E$ a locally convex, complete linear space. A strictly closed submodule $S$ of $C^*(X, E)$ is the intersection of all the maximal strictly closed submodules that contain it.

**Proof.** Let $S$ be any strictly closed submodule of $C^*(X, E)$. Fix $x$ in $X$ and let $M_x$ be the closure of $S(x)$ in $E$. Let $\mathfrak{M}(x, S)$ be the set of all maximal closed subspaces $M$ of $E$ which contain $M_x$. $M_x = \bigcap \{ M \mid M \in \mathfrak{M}(x, S) \}$ by the Hahn-Banach theorem for locally convex spaces. Clearly for each $x$ in $X$, $S$ is contained in $S_{x,M}$ when $M$ is in $\mathfrak{M}(x, S)$. Let $S' = \bigcap \{ S_{x,M} \mid x \in X, M \in \mathfrak{M}(x, S) \}$. $S'$ is a strictly closed submodule of $C^*(X, E)$ which contains $S$ and by Corollary 2 of Theorem 3 each function $f$ in $S'$ is in $S$ so that $S' = S$. 

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WEAK LIMITS OF POWERS OF A CONTRACTION IN HILBERT SPACE

S. R. FOGUEL

Let $T$ be an operator, on the Hilbert space $H$, with $\|T\| \leq 1$. Let

$$H_0 = \{ x \mid \text{weak lim } T^n x = 0 \}, \quad H_1 = H_0^\perp.$$

We shall use the facts, proved in \cite{1}, that

1. $x \in H_0$ if and only if $\lim (T^n x, x) = 0$ (Theorem 3.1).
2. On $H_1$ the operator $T$ is unitary (Theorem 1.1).

Given $x \in H$ let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. The purpose of this note is to find conditions, on the sequence $(T^n x, x)$, that will imply that $x_1$ is generated by eigenvectors of $T$. This is related to the notion of mixing for ergodic transformations.

**Theorem 1.** Let $y$ be in the subspace generated by $T^n x$, $T^{*n} x$, $n = 1, 2, \cdots$. If $\lim (T^n y, x) = 0$, then weak lim $T^n y = 0$.

**Proof.** Since $H_0$ and $H_1$ are invariant under $T$, it is enough to prove the theorem for the case when $x \in H_1$ (and thus also $y \in H_1$). If $n_i$ is any subsequence of the integers and $z = \text{weak lim } T^n y$, then $z$ is orthogonal to $T^k x$, $k = 0, \pm 1, \pm 2, \cdots$ (since $T$ is unitary on $H_1$). But $z$ belongs to the subspace generated by $T^\pm x$. Hence, $z = 0$ and, thus, weak lim $T^n y = 0$.

**Corollary.** Let $P(\lambda)$ be a polynomial whose roots of modulus one are $\lambda_1, \cdots, \lambda_k$. If $\lim (T^n P(T)x, x) = 0$, then $x = x_0 + x_1$ where: $x_0 \in H_0$, $x_1 = \sum_{i=1}^k z_i$, $Tz_i = \lambda_i x_i$.

**Proof.** Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. By Theorem 1,