CONCERNING THE COMMUTATOR
SUBGROUP OF A RING

W. E. BAXTER

This paper considers two independent results concerning \([A, A]\),
the commutator subgroup of an associative ring \(A\), and generated by
all elements \([a, b] = ab - ba\), where \(a\) and \(b\) are in \(A\). The first of these
results sharpens those of \([3]\), while the second uses the techniques
of \([6]\) to generalize \([1]\) and \([4]\). These results are stated as

**Theorem 1.** Let \(A\) be a simple associative ring; then either \(A\) is a
field or \([A, A]^2\), the subgroup generated by all products \(ab\) where \(a\) and
\(b\) are in \([A, A]\), is \(A\).

**Theorem 2.** Let \(A\) be an associative ring such that \([A, A]^7\), the sub-
ring generated by \([A, A]\), is \(A\) and let \(U\) be a Lie ideal of \([A, A]\), then
either \([[U, U], U]] = (0)\) or there exists a nontrivial (two-sided) ideal, \(R,
of A\) such that \(R \subseteq U^\prime\).

**Proof of Theorem 1.** Assume \(A\) is not 4-dimensional over \(Z\), its
center and a field of characteristic 2; if so, then a direct verification
shows that \([A, A]^2 = A\). Let \(x, y \in [A, A]\) and \(a \in A\), then \([x, y]a
= [x, ya] + y[a, x]\). Thus,

\[(xy - yx)A \subseteq [A, A] + [A, A]^3\]

for all \(x, y \in [A, A]\).

Now for any \(b \in A\), \(b[x, y]a = [b, [x, y]a] + [x, y]ab\) and hence,

\[A(xy - yx)A \subseteq [A, A] + [A, A]^2\]

for all \(x, y \in [A, A]\).

Therefore either (a) \([[A, A], [A, A]] = (0)\), or (b) \(A = [A, A] + [A, A]^2\).
(a) implies by \([4]\) and \([1]\) that \(A\) is a field, and (b) implies
\([A, A]^2 = A\) by the use of the following lemma.

**Lemma 1 (Herstein).** Let \(A\) be a simple associative ring, neither a
field nor 4-dimensional over its center, \(Z\), a field of characteristic 2. Then
\([A, A] \subseteq [A, A]^3\).

**Proof.** \([A, A]^2\) is obviously a Lie ideal of \(A\) and hence by \([3]\)
either is contained in \(Z\) or contains \([A, A]\). We now show that
\([A, A]^2 \subseteq Z\) leads to a contradiction. Let \(a, b, c \in A\); then \(u = [a, b][a, c]\) and
\(ua = [a, b][a, ca]\) are in \(Z\). Now if \(u \neq 0\), then the latter implies
that \(a \in Z\) and hence \(u = 0\), which is false. Thus, for all \(a, b, c \in A\),

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\([a, b][a, c]=0\). An easy verification shows that this leads to \(a \in Z\), a contradiction. Thus the desired conclusion.

**Proof of Theorem 2.** We assume that \([A, A]^\perp = A\). To prove the theorem we need the following lemma.

**Lemma 2.** Let \(U\) be a Lie ideal of \([A, A]\). Then \(I = I(U) = \{u \in U^- | au \in U^- \text{ for all } a \in A\}\) is an ideal of \(A\) with the property that it contains every ideal of \(A\) which is a subset of \(U^-\).

**Proof.** The latter statement is obvious from the definition of \(I\). It is also evident that \(I\) is a right ideal. Let \(b \in [A, A], a \in A,\) and \(u \in I.\) Then, \(b(ua) - (ua)b \in U^-\), and \(bu - ub \in U^-\) which implies that \([A, A]^\perp \subseteq I.\) Thus, for all \(n \geq 1, [A, A]^n[A, A]^n I \subseteq I\) and hence \(AI \subseteq I.\) So, \(I\) is an ideal of \(A.\) (The lemma also holds with \(U\) replacing \(U^-\) everywhere in the definition of \(I.\))

We are now in a position to prove Theorem 1. Suppose \([[U, U, U]] \neq \emptyset\); then there exists \(x \in [U, U], y \in U\) such that \(xy - yx \neq 0.\) Since \([[U, U], A] \subseteq [U, [U, A]] \subseteq U,\) we have \([x, y] \in U.\) Also, \([x, y]a = [x, ya] + y[x, a]\) for all \(a \in A.\) By the previous remark, \([x, ya]\) and \([x, a]\) are in \(U\) and thus \([x, y]a \in U^-\) for all \(a \in A.\) Thus, \(I \neq \emptyset,\) and by Lemma 2, the theorem is proved.

This theorem can be strengthened to Theorem 3 for certain rings using an argument similar to [3] and the following lemma.

**Lemma 3 [5].** If a ring \(A\) has no nonzero right ideal, \(J,\) with \(a^n = 0\) for all \(a \in J, n\) fixed, then \(A\) has a nonzero nilpotent (two-sided) ideal.

**Theorem 3.** Let \(A\) be a ring with no nilpotent ideals and such that \(2x = 0\) implies \(x = 0.\) Then either \(U^-\) contains a nontrivial ideal of \(A\) or \([U, U] \subseteq Z,\) the center of \(A.\)

**Proof.** We have seen that \([x, y] \in I\) for all \(x \in [U, U], y \in U.\) Thus, either \(U^-\) contains a nontrivial ideal of \(A\) or \([x, y] = 0\) for all \(x \in [U, U], y \in U.\) If the latter holds, then for all \(a \in A, [x, [x, a]] = 0.\) Setting \(a = bc\) and expanding the resulting expression, we obtain \(2[x, b][x, c] = 0\) for all \(b, c \in A\) which yields, using the hypothesis,

\[
\langle x, b \rangle^2 = 0 \quad \text{for all } x \in [U, U], b \in A. \tag{1}
\]

Suppose \([x, a] = 0, x \in [U, U],\) and for all \(a \in [A, A];\) then, since \([A, A]^\perp = A, x \in Z.\) Thus, assume that \(y = [x, b] \neq 0\) for some \(b \in [A, A].\) Then, \(y \in [U, U]\) and from (1) we have

\[
y^2 = 0 \quad \text{and} \quad [y, d]^2 = 0 \quad \text{for all } d \in A. \tag{2}
\]

Multiply (2) on the left by \(y\) and on the right by \(d\) and obtain \((yd)^2 = 0.\) Thus \(yA\) is a right ideal satisfying identity of Lemma 3. If \(yA \neq \emptyset,\)
then we have a contradiction, while \( yA = (0) \) implies \( A \) being simple, that \( y = 0 \), which also is a contradiction. Thus we have shown 
\[ U, U \subset Z. \]

This result indeed generalizes the work of \([1]\) and \([4]\).

Theorem 4. If \( A \) is simple (then \([A, A]^* = A\)) and \( U \) is a proper Lie ideal of \([A, A]\), then \( U \) is contained in the center of \( A \) except where \( A \) is of characteristic 2 and 4-dimensional over \( Z \), a field of characteristic 2.

Proof. Define \([U, U] = U^{(1)}\) and \(U^{(n+1)} = [U^{(n)}, U^{(n)}]\) for all \( n \geq 1 \). Then, since \( A \) is simple, it has no nonzero nilpotent ideals. Thus, except in characteristic 2, \([U, U] \subset Z\) or \( U^* = A \). If the former, then Theorems 7 and 9 of \([4]\), in the case not characteristic 3, and Lemma 3 of \([1]\) in this case implies \( U \subset Z \). Now, by these same results, if \( U^{(2)} \subset Z \), then \( U \subset Z \). Hence \( U^{(2)} = A \). Thus, by Lemma 9 of \([2]\) we have \([U^{(2)}, A] = [A, A]\), which contradicts \( U \) being proper. Lemma 1 of \([1]\) yields the result when \( A \) is of characteristic 2.

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References

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University of Delaware