SECOND ORDER LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. It was noted by Pinney [2] that the solution of the nonlinear differential equation \( y'' + p(x)y' + cy^{-3} = 0 \), \( c \) constant, can be written in the form \( y = (u_1^2 - u_2^2)^{1/2} \), where \( u_1(x), u_2(x) \) are appropriately chosen solutions of the linear equation \( u'' + p(x)u = 0 \). This result led Thomas [3] to ask: What equations of order \( n \) have general solutions expressible in the form \( y = F(u_1, \cdots, u_n) \), where \( u_1, \cdots, u_n \) constitute a variable set of solutions of a linear equation? Thomas answered this question when the underlying linear equation is of the first order, \( u' + pu = q \). He also gave the answer for homogeneous second order equations, \( u'' + pu' + qu = 0 \), when \( F \) depends only on one \( u \), or when \( F \) is homogeneous of nonzero degree in two \( u \)’s. Using the theory of passive partial differential equations, Herbst [1] removed these restrictions, obtaining the following general theorem.

Theorem 1. If \( u_1(x), u_2(x) \) are variable independent solutions with Wronskian \( w \) of the linear equation

\[(1.1) u'' = w'w^{-1}u' + qw,\]

where \( w(x), q(x) \) are arbitrarily prescribed functions, then the equation

\[(1.2) y'' = w'w^{-1}y' + f(y, y', w, q)\]

has general solution \( y = F(u_1, u_2) \) if, and only if, \( f \) has the form

\[(1.3) f = qZ(y) + A(y)(y')^2 + C(y)w^2,\]

where \( Z, A, C \) satisfy

\[(1.4) Z' - AZ = 1, \quad ZC + (3 - AZ)C = 0.\]

Herbst’s theorem determines the form of \( f \). In Herbst’s analysis, however, \( F \) is determined only as a solution of a system of four partial differential equations. The purpose of this paper is to give a simple characterization of \( F \). On the basis of information obtained a method is developed for the solution of (1.2) when \( f \) has form (1.3) and \( Z, A, C \) satisfy (1.4).

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2. Main result. For the determination of $F$ only a limited class of linear equations (1.1) is needed, namely, the class for which $w$ and $q$ are constants, $w \neq 0$. Our main result is as follows.

**Theorem 2.** Suppose that $f(y, y', w, q) \in C$ on a domain $R = \{(y, y', w, q) \mid m < y < M\}$, that

\[ \xi(y) = f(y, 0, 0, 1) \neq 0, \quad m < y < M, \]

and that $F(u_1, u_2) \in C'$ and $m < F < M$ on a domain $V$. Suppose further that for arbitrary constants $w \neq 0$ and $q$, if $u_1(x), u_2(x)$ are solutions of (1.1) with Wronskian $w$ such that $(u_1, u_2) \in V$ for $x$ on an interval $I$, then $y = F(u_1, u_2)$ satisfies (1.2) on $I$. Under these conditions, if $m < \eta < M$, then $F$ can be written

\[ F = \Phi(\omega^{1/2}), \quad (u_1, u_2) \in V, \]

where $\omega(u_1, u_2)$ is a homogeneous polynomial of degree 2, positive in $V$, and $\Phi$ is the inverse of

\[ \phi(y) = \exp \left\{ \int_{\eta}^{y} \xi^{-1}(t) \, dt \right\}, \quad m < y < M. \]

The proof is based on the uniqueness property of solutions of differential equations. Knowledge of the form (1.3) of $f$ is not required. We observe that in the Pinney case $\omega = u_1^2 - u_2^2$ and $\Phi(y) = y$.

**Proof.** We show first that

\[ L(u_1, u_2) = u_1 F_1(u_1, u_2) + u_2 F_2(u_1, u_2) = \xi(F(u_1, u_2)), \quad (u_1, u_2) \in V, \]

where $F_i = \partial F/\partial u_i$, $i = 1, 2$. Let $(v_1, v_2)$ be a point of $V$ other than the origin. Let $v'_1$, $v'_2$ be arbitrarily chosen so that $w = v'_1 v'_2 - v_2 v_1' \neq 0$. Let $u_1(x), u_2(x)$ be the solution on $(-\infty, \infty)$ of $u'' = u$ with initial values $u_1(0) = v_1, u'_1(0) = v'_1$. Then $u_1, u_2$ have Wronskian $w$, and $(u_1, u_2) \in V$ for $x$ on an interval $I$ containing $x = 0$. Hence, by hypothesis, $y = F(u_1, u_2)$ satisfies $y'' = f(y, y', w, 1)$ on $I$. Differentiating $y$ twice, and placing $x = 0$ in the final equation, we get

\[ v_1 F_1 + v_2 F_2 + F_{11}(v'_1)^2 + 2 F_{12} v'_1 v'_2 + F_{22}(v'_2)^2 = f(F, v_1 F_1 + v_2 F_2, w, 1), \]

where $F, F_i, F_{ij}$ are evaluated at $(v_1, v_2)$. With $v_1, v_2$ fixed we can let $v'_1, v'_2 \to 0$ in (2.5), to obtain $L(v_1, v_2) = f(F(v_1, v_2), 0, 0, 1)$. Hence, (2.4) holds save possibly at the origin. We observe, however, that the origin cannot be a point of $V$. Otherwise, by continuity, we would have $\xi(F(0, 0)) = L(0, 0) = 0$, contrary to (2.1).

Now let $P_0$: $(p_1, p_2)$ be a fixed point in $V$. Let $V_0$ be an open disk in
V having center at $P_0$. We prove that there exist constants $a_0, b_0, c_0$, not all zero, such that

$$
(2.6) \quad (b_0u_1 + c_0u_2)F_1(u_1, u_2) - (a_0u_1 + b_0u_2)F_2(u_1, u_2) = 0, \quad (u_1, u_2) \in V_0.
$$

By (2.1) and (2.4), the gradient of $F$ is different from zero at $P_0$. Accordingly, there are functions $\lambda_i(s), i = 1, 2$, of class $C^\infty$ on an interval $|s| < \sigma$, where $0 < \sigma$, such that $\lambda_i(0) = v_i^0$, $0 < (\lambda_i'(0))^2 + (\lambda_i''(0))^2$, and

$$
(2.7) \quad (\lambda_1, \lambda_2) \in V_0, \quad F(\lambda_1, \lambda_2) = F(v_1^0, v_2^0), \quad |s| < \sigma.
$$

Let $(v_1, v_2)$ be a second point of $V_0$ such that $w = v_1^0v_2 - v_2^0v_1 \neq 0$. Since $L(\lambda_1(s), \lambda_2(s)) \neq 0$, the equation

$$
(2.8) \quad \begin{pmatrix}
F_1(\lambda_1, \lambda_2) & F_2(\lambda_1, \lambda_2) \\
-\lambda_2 & \lambda_1
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
= \begin{pmatrix}
F_1(v_1^0, v_2^0) & F_2(v_1^0, v_2^0) \\
-0 & -v_2^0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
$$

admits for each $s$ on $(-\sigma, \sigma)$, a unique solution $z_s = z_s(s)$. From (2.4) and (2.7) we have

$$
(2.9) \quad L(\lambda_1, \lambda_2) = L(v_1^0, v_2^0), \quad |s| < \sigma.
$$

Hence,

$$
(2.10) \quad \begin{pmatrix}
z_1(s) \\
z_2(s)
\end{pmatrix}
= \begin{pmatrix}
\alpha(s) & \beta(s) \\
\gamma(s) & \delta(s)
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix},
$$

where

$$
(2.11) \quad L(v_1^0, v_2^0) \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 & -F_2(\lambda_1, \lambda_2) \\
\lambda_2 & F_2(\lambda_1, \lambda_2)
\end{pmatrix}
\begin{pmatrix}
F_1(v_1^0, v_2^0) & F_2(v_1^0, v_2^0) \\
-0 & -v_2^0
\end{pmatrix}.
$$

We note that $z_s = v_s$ for $s = 0$. Hence, for $|s|$ sufficiently small, $|s| < \sigma_1$, say, where $0 < \sigma_1 \leq \sigma$, we have $(z_1(s), z_2(s)) \in V_0$.

Suppose for the moment $s$ held fast on $(-\sigma_1, \sigma_1)$, and consider the linear functions $u_s(x) = u_s(x, s) = (1-x)\lambda_1 + xz_1, \quad 0 \leq x \leq 1$. We have $u_s(0) = \lambda_1, u_s(1) = \lambda_2$. Since $V_0$ is convex it follows that $(u_1, u_2) \in V_0, \quad 0 \leq x \leq 1$. Further, $u_1, u_2$ satisfy $u'' = 0$ and, by (2.8), have Wronskian

$$
\lambda_1(z_2 - \lambda_2) - \lambda_2(z_1 - \lambda_1) = \lambda_1z_2 - \lambda_2z_1 = v_1v_2 - v_2v_1.
$$

Hence, $y = y(x) = y(x, s) = F(u_1(x, s), u_2(x, s))$ satisfies

$$
(2.12) \quad y'' = f(y, y', w, 0), \quad 0 \leq x \leq 1.
$$

The equation (2.12) is independent of $s$. Referring to (2.7), (2.8), and (2.9), we verify that
\( y(0, s) = F(\lambda_1(s), \lambda_2(s)) = F(v_1^0, v_2^0), \)
\[
(2.13) \quad y'(0, s) = (\sigma_1 - \sigma_2)F_1(\lambda_1, \lambda_2) + (\sigma_3 - \sigma_2)F_2(\lambda_1, \lambda_2) = v_1F_1(v_1, v_2) + v_2F_2(v_1, v_2) - L(v_1, v_2).
\]

Thus, \( y(x, s), y'(x, s) \) have the same initial values, \( y(0, s), y'(0, s), \) independently of \( s. \) Since \( f \in C' \) on \( R, \) we conclude from the uniqueness property of solutions of differential equations that \( y(x, s) \) is independent of \( s \) on \( |s| < \sigma_1. \) Taking \( x = 1 \) we obtain

\[
(2.14) \quad F(z_1(s), z_2(s)) = F(z_1(0), z_2(0)) = F(v_1, v_2), \quad |s| < \sigma_1.
\]

We now differentiate in (2.14) and place \( s = 0. \) Using (2.10) we get
\[
(2.15) \quad (\alpha'(0)v_1 + \beta'(0)v_2)F_1(v_1, v_2) + (\gamma'(0)v_1 + \delta'(0)v_2)F_2(v_1, v_2) = 0.
\]

From (2.11) we obtain
\[
\begin{align*}
L\alpha'(0) &= \lambda_1'F_1 + v_2'\lambda_1 F_{12} + \lambda_2'F_{22}, \\
L\beta'(0) &= \lambda_2'F_2 - v_1'\lambda_1 F_{12} + \lambda_2'F_{22}, \\
L\gamma'(0) &= \lambda_1'F_1 - v_2'\lambda_1 F_{11} + \lambda_4'F_{12}, \\
L\delta'(0) &= \lambda_2'F_2 + v_1'\lambda_1 F_{11} + \lambda_4'F_{12},
\end{align*}
\]
where \( L, F_i, F_{ij} \) are evaluated at \((v_1^0, v_2^0), \) and \( \lambda_i \) at \( s = 0. \) By (2.7) and (2.9) we have for the same evaluations
\[
(2.16) \quad \lambda_1'F_1 + \lambda_4'F_2 = 0, \quad v_1'\lambda_1 F_{11} + \lambda_4'F_{12} + v_2'(\lambda_1 F_{12} + \lambda_4'F_{22}) = 0.
\]

Hence,
\[
(2.17) \quad \alpha'(0) + \delta'(0) = 0,
\]
\[
\begin{align*}
\nu_1^{(0)}\alpha'(0) + \nu_2^{(0)}\beta'(0) &= \lambda_1'(0), \\
\nu_1^{(0)}\gamma'(0) + \nu_2^{(0)}\delta'(0) &= \lambda_2'(0).
\end{align*}
\]

Now \( 0 < (\lambda_1'(0))^2 + (\lambda_4'(0))^2. \) Placing \( a_0 = -\gamma'(0), b_0 = \alpha'(0) = -\delta'(0), c_0 = \beta'(0), \) we conclude that \( a_0, b_0, c_0 \) are not all zero, and, using (2.15), that (2.6) holds at \((v_1, v_2). \) The only restriction on \((v_1, v_2) \) was that \( v_1^0v_2 - v_2^0v_1 \neq 0. \) Hence, by continuity, (2.6) holds in \( V_0. \)

We prove now that constants \( a, b, c, \) not all zero, can be chosen so that
\[
(2.18) \quad (bu_1 + cu_2)F_1(u_1, u_2) - (au_1 + bu_2)F_2(u_1, u_2) = 0, \quad (u_1, u_2) \in V.
\]

By our preceding analysis, if \( P \subseteq V \) and \( V_P \) is an open disk in \( V \) having center at \( P, \) there corresponds to \( P \) and \( V_P \) a set of constants \( a_P, b_P, \)
$c$, not all zero, such that (2.18) holds in $V_P$ with $a_P$, $b_P$, $c_P$ in place of $a$, $b$, $c$. We observe that since the gradient of $F$ is nonvanishing in $V$, if (2.18) holds with two different sets of constants on a domain $D \subset V$, these constants must be proportional. Hence, if we normalize the $a_P$, $b_P$, $c_P$, so that the first one which is not zero is one, then $a_P$, $b_P$, $c_P$ are uniquely determined. We show that the normalized $a_P$, $b_P$, $c_P$ are independent of $P$. Let $P_0$ be a fixed point in $V$, and let $P_1: (v_1^0, v_2^0)$ an arbitrary second point. Let $\Gamma: u_1 = \chi_1(t)$, $u_2 = \chi_2(t)$, $0 \leq t \leq 1$, $\chi_1(0) = v_1^0$, $\chi_1(1) = v_1^1$, be a Jordan arc in $V$ with endpoints at $P_0, P_1$. Let $E$ be the set of points $t$ on $[0,1]$ such that $a_P = a_{P_0}$, $b_P = b_{P_0}$, $c_P = c_{P_0}$, where $P$ has coordinates $(\chi_1(t), \chi_2(t))$. $E$ is not void, and has on $[0,1]$ a least upper bound $t'$, say. Utilizing our remark on the proportionality of constants for a domain $D$, we find first that $t' \in E$, and secondly that $t' = 1$. Thus $a_P = a_{P_0}$, $b_P = b_{P_0}$, $c_P = c_{P_0}$. Taking $a = a_{P_0}$, $b = b_{P_0}$, $c = c_{P_0}$, our conclusion relative to (2.18) follows.

The balance of the proof rests on (2.4) and (2.18). Let $a$, $b$, $c$, be constants, not all zero, for which (2.18) holds. Write $\omega(u_1, u_2) = au_1^2 + 2bu_1u_2 + cu_2^2$. From (2.4) and (2.18) we obtain

\[ \omega F_1 = (au_1 + bu_2)\xi(F), \quad \omega F_2 = (bu_1 + cu_2)\xi(F) \]

for $(u_1, u_2) \in V$. We show first that $\omega \neq 0$ on $V$. Suppose, if possible, that $(v_1, v_2) \in V$ and $\omega(v_1, v_2) = 0$. Since $\xi(F) \neq 0$, it then follows that $av_1 + bv_2 = 0$, $bv_1 + cv_2 = 0$. Since $V$ does not contain the origin, we then have $b^2 - ac = 0$. Not both $a$ and $c$ vanish, and we can assume without loss of generality that $a \neq 0$. In this case we have $\omega = (au_1 + bu_2)^2/a$ in $V$, and accordingly, by (2.19), $(au_1 + bu_2)^2 F_1 = a(au_1 + bu_2)\xi(F)$ in $V$. From this relation we obtain $(au_1 + bu_2)F_1 = a\xi(F)$ in $V$ except possibly on the line $au_1 + bu_2 = 0$. By continuity we then have $\xi(F) = (au_1 + bu_2)F_1/a = 0$ at $(v_1, v_2)$, which is a contradiction. Thus, $\omega \neq 0$ in $V$.

From (2.19) we have

\[ dF = (1/2)\xi(F)\omega^{-1}d\omega, \quad (u_1, u_2) \in V. \]

Let $\eta$ be arbitrary on $(m, M)$. Motivated by (2.20) we define $\phi$ by (2.3). Then $\phi \in C'$ and $\phi' \neq 0$ on $(m, M)$. From (2.20) we get

\[ \phi^{-1}F d\phi(F) = (1/2)\omega^{-1}d\omega, \quad (u_1, u_2) \in V. \]

Since $\omega \neq 0$, we can normalize $a$, $b$, $c$ so that at an arbitrary point $(v_1, v_2)$ in $V$ we have $0 < \omega$ and $\phi(F(v_1, v_2)) = \omega^{1/2}(v_1, v_2)$. The $\omega$ of our theorem is then determined. From (2.21) we obtain $\phi(F) = \omega^{1/2}$ in $V$. But $\phi$ has an inverse $\Phi$. Thus, $F = \Phi(\omega^{1/2})$ in $V$. This completes the proof.
3. Solutions of (1.2). We now consider briefly the integration of (1.2) when \( f \) has the form (1.3) and \( Z, A, C \) satisfy (1.4). We assume that \( A \in C^0, Z \in C', C \in C' \), and that the equations (1.4) hold on an interval \( m < y < M \). We note that the first equation in (1.4) implies that \( Z \) vanishes at most once on \( (m, M) \). Also, the two equations imply that \( Y = Z^2C \) satisfies \( Y' = 4AY \). Hence, \( ZC \neq 0 \) or \( ZC = 0 \) on \( (m, M) \). It suffices then to treat two cases: Case I. \( Z \neq 0 \) on \( (m, M) \); Case II. \( C = 0 \) and \( Z = 0 \) at some one point of \( (m, M) \).

Case I. For the case \( Z \neq 0 \), Theorem 2 is directly applicable. For \( f \) in the form (1.3), \( f(y, 0, 0, 1) = Z(y) \). We then define \( \phi \) by (2.3) with \( \xi = Z \) and \( \eta \) arbitrary on \( (m, M) \). We may observe that because of (1.4) \( \phi \) can be written

\[
(3.1) \quad \phi(y) = Z^{-1}(\eta)Z(y) \exp \left( - \int_{r}^{y} A(t) \, dt \right), \quad m < y < M.
\]

Denoting by \( \Phi \) the inverse of \( \phi \), we have \( \mathcal{F}(i) \subseteq C'' \), \( \Phi \neq 0 \) on \( g < t < G \), where \( (g, G) \) is the range of \( \phi \). Plainly, \( 0 \leq g \). It remains to determine \( \omega \). Let \( \omega(u_1, u_2) = au_1^2 + 2bu_1u_2 + cu_2^2 \) be an arbitrary homogeneous polynomial of degree 2 which is positive at some point in the plane. Put \( \Delta = b^2 - ac \), and let \( V \) be the (or a) domain determined by \( 0 < \omega \), \( g < \omega^{1/2} < G \). Suppose that \( w(x), g(x) \) are arbitrary functions satisfying \( w \in C' \), \( w \neq 0 \), \( g \in C^0 \) on an interval \( J \). Let \( x_0 \) be a point of \( J \) and let \( y_0, y'_0 \) be arbitrary initial values, where \( m < y_0 < M \). Evidently we can find \( (v_1, v_2) \) in \( V \) such that \( \omega^{1/2} = \omega(v_1, v_2) = \phi(y_0) \). Furthermore, \( v'_1, v'_2 \) can be determined so that \( wv'_2 - v_1v'_2 = w(x_0) \), \( \mathcal{F}(\omega^{1/2}) [v_1, v'_1] \omega^{1/2} = y'_1 \), where \( [v_1, v'_1] = av_1v'_2 + bv_1v'_2 + bv_2v'_2 + cv_2v'_2 \). Denote by \( u_1(x), u_2(x) \) the solutions of (1.1) on \( J \) with initial values \( u_1(x_0) = v_1, \ u'_1(x_0) = v'_1 \). Then \( u_1(x), u_2(x) \) have Wronskian \( w \) and \( (u_1, u_2) \in V \) for \( x \) in an interval \( I \) containing \( x_0 \). Defining \( y \) by (1/2) in \( \omega = \int_{x}^{y} Z^{-1}(\eta) \, dt \), and making use of (1.1), the first equation in (1.4) and the identity

\[
(3.2) \quad [u_1, u_2][u'_1, u'_2] = [u_1, u'_2]^2 - \Delta(u_1u'_2 - u_2u'_2),
\]

we find that \( y = y(x) = \Phi(\omega^{1/2}(u_1(x), u_2(x))) \) satisfies

\[
(3.3) \quad y'' = w'(x)w^{-1}(x)y' + g(x)Z(y) + A(y)(y')^2 - \Delta Z(y)\omega^{-2}w^2(x).
\]

But

\[
(3.4) \quad C(y) = C(\eta)Z^2(\eta)Z^{-2}(y) \exp \left( 4 \int_{t}^{y} A(t) \, dt \right) = C(\eta)Z^{-1}(\eta)Z(y)\omega^{-2}.
\]

Hence, \( y \) satisfies (1.2) on \( I \) if \( \Delta = -C(\eta)Z^{-1}(\eta) \). We have, further, \( y(x_0) = \Phi(\omega^{1/2}) = y_0, \ y'(x_0) = \Phi'(\omega^{1/2}) [v_1, v'_1] \omega^{1/2} = y'_0 \). Accordingly,
for Case I, if $\Delta = -C(\eta)Z^{-1}(\eta)$, then for arbitrary $w(x)$, $q(x)$ having the properties prescribed above, $y = \Phi(\omega^{1/2}(u_1, u_2))$, together with solutions $u_1(x)$, $u_2(x)$ of (1.1) having Wronskian $w$, and such that $(u_1(x), u_2(x)) \in V$ provide the general solution of (1.2).

Case II. If $Z$ vanishes at a point of $(m, M)$, then the definition (2.3) is inappropriate. However, if $\eta$ is chosen so that $Z(\eta) \neq 0$, then (3.1) is applicable. Defining $\phi$ by (3.1), and using the first equation in (1.4), we find that

$$\phi'(y) = Z^{-1}(\eta) \exp \left( - \int_{\eta}^{y} A(t) \, dt \right), \quad m < y < M. \tag{3.5}$$

Thus, $\phi \in C''$ and $\phi' \neq 0$ on $(m, M)$. Denoting by $\Phi$ the inverse of $\phi$, we have again $\Phi(t) \in C''$, $\Phi' \neq 0$ on $g < t < G$, where $(g, G)$ is the range of $\phi$. In this case, $g < 0 < G$. Let $\omega(u_1, u_2) = au_1 + bu_2$ be an arbitrary nontrivial linear function. Determine $V$ by $g < \omega < G$. Introducing $w, q, u_1, u_2$ as in the preceding paragraph, we find that $y = \Phi(au_1 + bu_2)$ satisfies

$$y'' = w'w^{-1}y' + qZ(y) + A(y)(y')^2. \tag{3.6}$$

Since in the case under consideration we have $C = 0$, we see that $y = \Phi(\omega(u_1, u_2))$, together with appropriate solutions of (1.1), provide the general solution (1.2) for Case II.

We may observe that canonical forms for the solution of (1.2) are

$$y = \Phi([u_1 + u_2]^{1/2}), \quad y = \Phi([u_1 - u_2]^{1/2}), \quad y = \Phi(u_1),$$

where $u_1, u_2$ are solutions of (1.1).

REFERENCES


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