SOME HILBERT SPACE EXTREMAL PROBLEMS

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In this note we shall generalize in several ways the following theorem of Philip Davis:

Let $l^2$ be a real or complex sequential Hilbert space. Fix $\{h_j\}_{j=0}^\infty$ in $l^2$, a positive real number $M$, and a sequence $\{k_j\}_{j=0}^\infty$ such that $k_j > 0$ for all $j$ and $\sum_{j=0}^\infty k_j^{-1}$ is finite. Suppose that $\sum_{j=0}^\infty k_j |h_j|^2 > M^2$.

Then there exists a unique element $\{e_j\} \in l^2$ such that the infimum in

$$(a') \inf \left\{ \sum_{j=0}^\infty |h_j - a_j|^2 : \{a_j\} \in l^2, \sum_{j=0}^\infty k_j |a_j|^2 \leq M^2 \right\}$$

is achieved. This is given by $e_j = (1 + \lambda k_j)^{-1}h_j$, where $\lambda$ is the unique positive root of $\sum_{j=0}^\infty k_j(1 + \lambda k_j)^{-1}|h_j|^2 = M^2$.

Davis proved the result by first considering the extremal problem for $n$ dimensional Euclidean space $R^n$. In this case the problem has the following geometric interpretation:

$$(a'') \text{Let } E \text{ be the hyperellipse } \{(a_1, \ldots, a_n): \sum_{j=0}^n k_j a_j^2 \leq M^2\} \text{ in } R^n \text{ and suppose that } h = (h_1, \ldots, h_n) \text{ is not in } E. \text{ Find the vector } e \text{ in } E \text{ such that the distance from } h \text{ to } e \text{ is a minimum.}$$

This $R^n$ problem is readily handled by Lagrange multipliers. The growth condition on the $k_j$ permits the use of finite dimensional approximations to deduce the more general result. See [1] or [2, p. 204].

Our first generalization of Davis' theorem is

**Theorem 1.** Let $\mathcal{H}$ be a real or complex Hilbert space, and suppose $h \in \mathcal{H}$. Suppose $A$ is a closed linear operator on $\mathcal{H}$ to $\mathcal{H}$ whose domain $\mathcal{D}(A)$ is dense in $\mathcal{H}$. Fix $M > 0$.

Then there is a unique element $e$ in $\mathcal{H}$ such that the infimum in

$$(\alpha) \inf \{|h - a| : \|Aa\| \leq M\}$$

is achieved. $e = h$ if and only if $\|Ah\| \leq M$. If $e \neq h$, then $e = (I + \lambda A^*A)^{-1}h$, where $\lambda$ is the unique positive solution of $\|A(I + \lambda A^*A)^{-1}h\| = M$.

It is clear that $(a')$ is a special case of $(\alpha)$. If $A$ is specialized so that $A^{-1}$ exists and is a Schmidt-Hilbert operator, then Davis' theorem and Theorem 1 are in fact equivalent.

Our proof of Theorem 1 is carved up into three lemmas. Through-

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out we assume that $\mathfrak{C}, h, A$ and $M$ are as stated in the hypotheses of Theorem 1.

**Lemma 1.** There exists a unique extremal element $e$ minimizing $(\alpha)$.

**Proof.** $\{a: \|Aa\| \leq M\}$ is clearly a convex set, we will show that it is closed and this will imply Lemma 1. Suppose $a_n \rightarrow a$ and $\|Aa_n\| \leq M$. Extract a subsequence $\{b_n\}$ of $\{a_n\}$ such that $Ab_n$ converges weakly, say, to $b$. Thus for any $c \in \mathcal{D}(A^*)$

$$\langle b, c \rangle = \lim_{n \to \infty} \langle Ab_n, c \rangle = \lim_{n \to \infty} \langle b_n, A^*c \rangle = \langle a, A^*c \rangle.$$ 

Thus $A^*a = b$, and since $A$ is closed, $Aa = b$. $\|Aa\| \leq \liminf_{n \to \infty} \|Aa_n\|$ $\leq M$, so $a$ is in the convex set.

The following lemma is basic for our proof, and despite appearances, is quite geometrical in nature. Suppose $\mathfrak{C} = \mathbb{R}^n$ and consider the situation of $(\alpha^\prime)$. $(i)$ states that if $a$ is tangent to the boundary of $E$ at $e$, then $h-e$ is perpendicular to $a$. $(ii)$ shows that the angle between $h-e$ and $e$ satisfies $-\pi/2 \leq \theta \leq \pi/2$.

**Lemma 2.** $(i)$ If $a \in \mathcal{D}(A)$ and $\langle Ae, Aa \rangle = 0$, then $\langle h-e, a \rangle = 0$. In particular, if $Ae = 0$ then $e = h$.

$(ii)$ $\langle h-e, e \rangle \geq 0$.

**Proof.** We shall give the proof for complex Hilbert spaces only as the real case can be then handled by simple modifications. Assume the hypotheses of $(i)$. With no loss in generality we may assume that $\|Aa\| \leq M$. Let $\mu$ and $\nu$ be complex numbers such that $|\mu|^2 + |\nu|^2 \leq 1$. Then $\|A(\mu e + \nu a)\| \leq M$. Thus $0 \leq \|h - \mu e - \nu a\|^2 - \|h-e\|^2$, so $0 \leq 2 Re \left[(1-\mu^*)\langle h-e, e \rangle - 2 Re[\nu^*\langle h-e, a \rangle] + \|\mu e - \nu a\|_2 \right]$, where $*$ is the complex conjugation operation. We derive $(i)$ and $(ii)$ by selecting suitable $\mu$ and $\nu$.

To prove $(i)$ choose $\mu = 1-t$, $0 < t < 1$, and $\nu = (2t-t^2)^{1/2} \sgn \langle h-e, a \rangle$. Then $0 \leq -2(2t-t^2)^{1/2} |\langle h-e, a \rangle| + O(t)$ as $t \to 0+$, and this implies $(i)$.

Next we prove $(ii)$. Pick $\theta$ so $\cos \theta > 0$, and set $\mu^* = 1-e^{i\theta t}$, $0 < t$, $t$ small, $a = 0$, and $\nu = 0$. Then $0 \leq 2t Re \left[e^{i\theta}\langle h-e, e \rangle \right] + O(t^2)$ as $t \to 0+$, so $Re \left[e^{i\theta}\langle h-e, e \rangle \right] \geq 0$. This implies $(ii)$.

**Lemma 3.** Either $e = h$ or there exists a unique positive number $\lambda$ such that $(I + \lambda A^*A)e = h$.

**Proof.** If $Ae = 0$ then Lemma 2(i) guarantees that $e = h$. Thus we may assume that $Ae \neq 0$. Let $b$ be in $\mathcal{D}(A^*A) \subset \mathcal{D}(A)$ and consider $a = \|Ae\|^2 b - (Ab, Ae)e$. Then $\langle Ae, Aa \rangle = 0$, so we may invoke Lemma 2(i) to deduce that $\langle h-e, a \rangle = 0$. It follows that $\langle h-e, b \rangle = \lambda \langle Ae, Ab \rangle$. 

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where \( \lambda = \| A e \|^{-2} \langle h - e, e \rangle \leq 0 \). If \( \lambda = 0 \) then \( h - e \) is perpendicular to \( \mathcal{D}(A^*A) \), so \( h = e \). Assume then that \( \lambda > 0 \). We have \( \langle h, b \rangle = \langle e, (I + \lambda A^*A)b \rangle \) for all \( b \in \mathcal{D}(A^*A) \), so \( (I + \lambda A^*A)e = h \). The condition \( A e \neq 0 \) implies that \( \lambda \) is unique.

Proof of Theorem 1. Suppose it is not the case that \( \| A h \| \leq M \), so \( e \neq h \). Then there exists \( \lambda > 0 \) such that

\[
\| h - e \| = \| h - (I + \lambda A^*A)^{-1}h \| = \| \lambda A^*A(I + \lambda A^*A)^{-1}h \|.
\]

This last expression is a strictly increasing function of \( \lambda \). Now, \( \| A e \|^2 = \langle A^*A(I + \lambda A^*A)^{-2}h, h \rangle \). This is a strictly decreasing function of \( \lambda \) that approaches a number greater than \( M \) or diverges to infinity as \( \lambda \to 0^+ \). Thus necessarily \( \| A e \| = M \).

An alternate formulation of Davis’ theorem considers the extremal problem

\[
\inf \left\{ \sum_{j=0}^{\infty} b_j^2 : \sum_{j=0}^{\infty} \| b_j \|^2 \leq M \right\}.
\]

This problem is equivalent to \( \langle \alpha' \rangle \) by an obvious change of variables. We turn now to a generalization of \( \langle \beta' \rangle \).

Theorem 2. Let \( \mathcal{H} \) be a complex Hilbert space, \( h \) an element of \( \mathcal{H} \) and \( M \) a positive real number. Suppose \( B \) is a closed densely defined linear operator on \( \mathcal{H} \) to \( \mathcal{H} \) with polar decomposition \( B = W|B| \), so \( WW^* \) is the projection operator on the closure of the range of \( B \).

Then there is a unique element \( B_f \) in \( \mathcal{H} \) such that the infimum in

\[
\inf \{ \| h - Bb \| : \| b \| \leq M \} = \beta(h)
\]

is achieved. \( B_f = WW^*h \) if and only if there exists \( b \) with \( \| b \| \leq M \) and \( WW^*h = Bb \). If \( B_f \neq WW^*h \), then

\[
B_f = W \left( |B|^2 (|B|^2 + \lambda I)^{-1} \right) W^*h,
\]

where \( \lambda \) is the unique positive solution of

\[
\| B \left( |B|^2 + \lambda I \right)^{-1} W^*h \| = M.
\]

Let \( B \) and \( W \) be as in Theorem 2. First we prove

Lemma 4. There is a unique extremal element \( B_f \) minimizing \( \langle \beta \rangle \).

Proof. \( \{ Bb : \| b \| \leq M \} \) is a convex set. The same technique used in the proof of Lemma 1 shows that it is closed.

We fix \( B_f \) as the above extremal element. \( |B| = (B^*B)^{1/2} \) is a self-adjoint operator such that \( \mathcal{D}(|B|) = \mathcal{D}(B) \) and \( \| Bb \| = \| |B| b \| \) for
all $b \in \mathcal{D}(B)$. $B$ and $|B|$ have the same nullspace, $(I - W^*W)\mathcal{H}$. (See [3, pp. 304–308] for a discussion of the polar decomposition.) In considering problem (β) it suffices to consider $b$ orthogonal to this nullspace, that is, $b$ in $W^*W\mathcal{H}$. There is a self-adjoint operator $A$ such that $A |B| b = W^*Wb$ for all $b$ in $\mathcal{D}(B)$; $A$ is the inverse of $|B|$ relative to the Hilbert space $W^*W\mathcal{H}$. $W^*W$ is the projection operator on the closure of the range of $|B|$.

**Lemma 5.**

$$\beta^2(h) = \|h\|^2 - \|W^*h\|^2 + \inf\{\|W^*h - a\|^2 : \|Aa\| \leq M\}.$$  

This last infimum is uniquely achieved by an element $e$ in $\mathcal{D}(A) \subset W^*W\mathcal{H}$. $Bf = We$.

**Proof.** If $b$ is in $\mathcal{D}(A)$, then

$$\|h - Bb\|^2 = \|h - W |B| b\|^2$$

$$= \|h\|^2 - \langle |B| b, W^*h \rangle - \langle W^*h, |B| b \rangle + \| |B| b\|^2$$

$$= \|h\|^2 - \|W^*h\|^2 + \|W^*h - |B| b\|^2.$$  

Thus

$$\beta^2(h) = \|h\|^2 - \|W^*h\|^2$$

$$+ \inf\{\|W^*h - |B| b\|^2 : \|b\| \leq M, b \in W^*W\mathcal{H}\}$$

$$= \|h\|^2 - \|W^*h\|^2 + \inf\{\|W^*h - a\|^2 : \|Aa\| \leq M\}.$$  

An application of Theorem 1 shows that this last extremal problem has a unique solution $e$. Necessarily $|B| f = e$, so $Bf = We$.

**Proof of Theorem 2.** We list a quadruple of equivalent statements: $Bf = WW^*h$; $e = W^*h$; $\|A W^*h\| \leq M$; there exists $b$ with $\|b\| \leq M$ and $WW^*h = Bb$. The first two are equivalent since $W$ is a partial isometry, the next are equivalent by Theorem 1 as applied to the situation of Lemma 5. The final equivalence is a consequence of the definition of $A$.

Next suppose that $e \neq W^*h$. Then by applying Theorem 1 to the problem in Lemma 5 we see that the infimum in (β) is achieved by $Bf = We = W(I + \lambda A^2)^{-1}W^*h = W |B| (|B|^2 + \lambda I)^{-1}W^*h$, where $\lambda$ is the unique positive solution of

$$M = \|A(I + \lambda A^2)^{-1}W^*h\| = \| |B| (|B|^2 + \lambda I)^{-1}W^*h\|.$$  

When applying Theorem 2 it is sometimes convenient to note that (2.1) and (2.2) can be simplified if $h$ is in $\mathcal{D}(B^*)$. (2.1) becomes

$$(2.1)'$$

$$Bf = B(B^*B + \lambda I)^{-1}B^*h$$
and (2.2) becomes

\[(2.2)\quad \left\| (B^*B + \lambda I)^{-1}B^*h \right\| = M.\]

As an elementary application we have

**Theorem 3.** Let \( D(D) \) be the set of all \( b \) in the complex Lebesgue space \( \mathcal{C} = L^2(-\infty, \infty) \) such that \( b \) is locally absolutely continuous on \( (-\infty, \infty) \) and whose derivative \( b' \) is in \( \mathcal{C} \). \( D \) sends any \( b \) in \( D(D) \) into \( ib' \). Let \( h \) be in \( L^2(-\infty, \infty) \) and \( M \) be a positive real number.

Then the extremal problem \( \inf \{ ||h - D\beta|| : ||\beta|| \leq M \} \) has a unique extremal element \( Df \). Either \( h = Df \) or

\[(3.1)\quad (Df)(x) = h(x) - \frac{\omega}{2} \int_{-\infty}^{\infty} h(t) \exp(-\omega |x - t|) dt\]

for some real positive number \( \omega \).

**Proof.** As shown in Stone [4, pp. 441–445], \( D = \mathfrak{F}\mathfrak{F}^* \), where \( \mathfrak{F} \) is the Fourier-Plancherel transform and \( \mathbb{S} \) is the self-adjoint operator that sends \( f(x) \) into \( x f(x) \). Then by Theorem 2 either \( h = Df \), or, by

\[(2.1)\quad Df = D^2(D^2 + \omega^2 I)^{-1}h = h - \omega^2 \mathfrak{F}(\mathbb{S}^2 + \omega^2 I)^{-1}\mathfrak{F}^*h, \] 

where \( \omega > 0 \). Thus (3.1) holds.

**References**