SOME HILBERT SPACE EXTREMAL PROBLEMS

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In this note we shall generalize in several ways the following theorem of Philip Davis:

Let \( l^2 \) be a real or complex sequential Hilbert space. Fix \( \{ h_j \} \) in \( l^2 \), a positive real number \( M \), and a sequence \( \{ k_j \} \) such that \( k_j > 0 \) for all \( j \) and \( \sum_{j=0}^{\infty} k_j^{-1} \) is finite. Suppose that \( \sum_{j=0}^{\infty} k_j |h_j|^2 > M^2 \).

Then there exists a unique element \( \{ e_j \} \in l^2 \) such that the infimum in

\[
(\alpha') \quad \inf \left\{ \sum_{j=0}^{\infty} |h_j - a_j|^2 : \{ a_j \} \in l^2, \sum_{j=0}^{\infty} k_j |a_j|^2 \leq M^2 \right\}
\]

is achieved. This is given by \( e_j = (1 + \lambda k_j)^{-1} h_j \), where \( \lambda \) is the unique positive root of \( \sum_{j=0}^{\infty} k_j (1 + \lambda k_j)^{-1} |h_j|^2 = M^2 \).

Davis proved the result by first considering the extremal problem for \( n \) dimensional Euclidean space \( \mathbb{R}^n \). In this case the problem has the following geometric interpretation:

(\( \alpha'' \)) Let \( E \) be the hyperellipse \( \{ (a_1, \ldots, a_n) : \sum_{j=0}^{n} k_j a_j^2 \leq M^2 \} \) in \( \mathbb{R}^n \) and suppose that \( h = (h_1, \ldots, h_n) \) is not in \( E \). Find the vector \( e \) in \( E \) such that the distance from \( h \) to \( e \) is a minimum.

This \( \mathbb{R}^n \) problem is readily handled by Lagrange multipliers. The growth condition on the \( k_j \) permits the use of finite dimensional approximations to deduce the more general result. See [1] or [2, p. 204].

Our first generalization of Davis' theorem is

**Theorem 1.** Let \( \mathcal{H} \) be a real or complex Hilbert space, and suppose \( h \in \mathcal{H} \). Suppose \( A \) is a closed linear operator on \( \mathcal{H} \) to \( \mathcal{H} \) whose domain \( \mathcal{D}(A) \) is dense in \( \mathcal{H} \). Fix \( M > 0 \).

Then there exists a unique element \( e \) in \( \mathcal{H} \) such that the infimum in

\[
(\alpha) \quad \inf \{ ||h - a|| : ||Aa|| \leq M \}
\]

is achieved. \( e = h \) if and only if \( ||Ah|| \leq M \). If \( e \neq h \), then \( e = (I + \lambda A^*A)^{-1} h \), where \( \lambda \) is the unique positive solution of \( ||A(I + \lambda A^*A)^{-1}h|| = M \).

It is clear that (\( \alpha' \)) is a special case of (\( \alpha \)). If \( A \) is specialized so that \( A^{-1} \) exists and is a Schmidt-Hilbert operator, then Davis' theorem and Theorem 1 are in fact equivalent.

Our proof of Theorem 1 is carved up into three lemmas. Through-
LEMMA 1. There exists a unique extremal element $e$ minimizing $(\alpha)$.

Proof. $\{a: \|Aa\| \leq M\}$ is clearly a convex set, we will show that it is closed and this will imply Lemma 1. Suppose $a_n \to a$ and $\|Aa_n\| \leq M$. Extract a subsequence $\{b_n\}$ of $\{a_n\}$ such that $A b_n$ converges weakly, say, to $b$. Thus for any $c \in \mathcal{D}(A^*)$

$$\langle b, c \rangle = \lim_{n \to \infty} \langle A b_n, c \rangle = \lim_{n \to \infty} \langle b_n, A^*c \rangle = \langle a, A^*c \rangle.$$
where \( \lambda = \| A e \|^{-1} h - e, e \| \geq 0 \). If \( \lambda = 0 \) then \( h - e \) is perpendicular to \( \mathcal{D}(A^*A) \), so \( h = e \). Assume then that \( \lambda > 0 \). We have \( \langle h, b \rangle = \langle e, (I + \lambda A^*A)b \rangle \) for all \( b \in \mathcal{D}(A^*A) \), so \( (I + \lambda A^*A)e = h \). The condition \( Ae \neq 0 \) implies that \( \lambda \) is unique.

**Proof of Theorem 1.** Suppose it is not the case that \( \| Ae \| \leq M \), so \( e \neq h \). Then there exists \( \lambda > 0 \) such that

\[
\| h - e \| = \| h - (I + \lambda A^*A)^{-1} h \| = \| \lambda A^*A (I + \lambda A^*A)^{-1} h \|.
\]

This last expression is a strictly increasing function of \( \lambda \). Now, \( \| A e \|^2 = \langle A^*A (I + \lambda A^*A)^{-2} h, h \rangle \). This is a strictly decreasing function of \( \lambda \) that approaches a number greater than \( M \) or diverges to infinity as \( \lambda \to 0+ \). Thus necessarily \( \| Ae \| = M \).

An alternate formulation of Davis' theorem considers the extremal problem

\[
\beta') \quad \inf \left\{ \sum_{j=0}^{\infty} | h_j - k^j b_j |^2 : \sum_{j=0}^{\infty} | b_j |^2 \leq M \right\}.
\]

This problem is equivalent to \( \alpha' \) by an obvious change of variables. We turn now to a generalization of \( \beta' \).

**Theorem 2.** Let \( \mathcal{H} \) be a complex Hilbert space, \( h \) an element of \( \mathcal{H} \) and \( M \) a positive real number. Suppose \( B \) is a closed densely defined linear operator on \( \mathcal{H} \) to \( \mathcal{H} \) with polar decomposition \( B = W | B | \), so \( WW^* \) is the projection operator on the closure of the range of \( B \).

Then there is a unique element \( Bf \) in \( \mathcal{H} \) such that the infimum in

\[
\beta \quad \inf \{ \| h - Bb \| : \| b \| \leq M \} = \beta(h)
\]

is achieved. \( Bf = WW^*h \) if and only if there exists \( b \) with \( \| b \| \leq M \) and \( WW^*h = Bb \). If \( Bf \neq WW^*h \), then

\[
(2.1) \quad Bf = W | B | ( | B |^2 + \lambda I)^{-1} W^*h,
\]

where \( \lambda \) is the unique positive solution of

\[
(2.2) \quad \| | B | ( | B |^2 + \lambda I)^{-1} W^*h \| = M.
\]

Let \( B \) and \( W \) be as in Theorem 2. First we prove

**Lemma 4.** There is a unique extremal element \( Bf \) minimizing \( \beta \).

**Proof.** \( \{ Bb : \| b \| \leq M \} \) is a convex set. The same technique used in the proof of Lemma 1 shows that it is closed.

We fix \( Bf \) as the above extremal element. \( | B | = (B^*B)^{1/2} \) is a self-adjoint operator such that \( \mathcal{D}(B) = \mathcal{D}(B^*B) \) and \( \| Bb \| = \| | B | b \| \) for
all \( b \in \mathcal{D}(B) \). \( B \) and \( | B | \) have the same nullspace, \((I - W^{*}W)\mathfrak{C}\). (See [3, pp. 304–308] for a discussion of the polar decomposition.) In considering problem (\( \beta \)) it suffices to consider \( b \) orthogonal to this nullspace, that is, \( b \) in \( W^{*}W\mathfrak{C} \). There is a self-adjoint operator \( A \) such that \( A | B | b = W^{*}Wb \) for all \( b \) in \( \mathcal{D}(B) \); \( A \) is the inverse of \( | B | \) relative to the Hilbert space \( W^{*}W\mathfrak{C} \). \( W^{*}W \) is the projection operator on the closure of the range of \( | B | \).

**Lemma 5.**

\[
\beta^{2}(h) = \|h\|^{2} - \|W^{*}h\|^{2} + \inf\{\|W^{*}h - a\|^{2}; \|Aa\| \leq M\}.
\]

This last infimum is uniquely achieved by an element \( e \) in \( \mathcal{D}(A) \subset W^{*}W\mathfrak{C} \). \( Bf = We \).

**Proof.** If \( b \) is in \( \mathcal{D}(A) \), then

\[
\|h - Bb\|^{2} = \|h - W|B||b\|^{2}
\]

\[
= \|h\|^{2} - (|B|b, W^{*}h) - (W^{*}h, |B|b) + \| |B|b\|^{2}
\]

\[
= \|h\|^{2} - \|W^{*}h\|^{2} + \|W^{*}h - |B|b\|^{2}.
\]

Thus

\[
\beta^{2}(h) = \|h\|^{2} - \|W^{*}h\|^{2} + \inf\{\|W^{*}h - |B|b\|^{2}; \|b\| \leq M, b \in W^{*}W\mathfrak{C}\}
\]

\[
= \|h\|^{2} - \|W^{*}h\|^{2} + \inf\{\|W^{*}h - a\|^{2}; \|Aa\| \leq M\}.
\]

An application of Theorem 1 shows that this last extremal problem has a unique solution \( e \). Necessarily \( |B|e = e \), so \( Bf = We \).

**Proof of Theorem 2.** We list a quadruple of equivalent statements: \( Bf = WW^{*}h \); \( e = W^{*}h \); \( \|AW^{*}h\| \leq M \); there exists \( b \) with \( \|b\| \leq M \) and \( WW^{*}h = Bb \). The first two are equivalent since \( W \) is a partial isometry, the next are equivalent by Theorem 1 as applied to the situation of Lemma 5. The final equivalence is a consequence of the definition of \( A \).

Next suppose that \( e \neq W^{*}h \). Then by applying Theorem 1 to the problem in Lemma 5 we see that the infimum in (\( \beta \)) is achieved by \( Bf = We = W(1 + \lambda A^{2})^{-1}W^{*}h = W|B|^{2}(1 + \lambda I)^{-1}W^{*}h \), where \( \lambda \) is the unique positive solution of

\[
M = \|A(1 + \lambda A^{2})^{-1}W^{*}h\| = \| |B|^{2}(1 + \lambda I)^{-1}W^{*}h\|.
\]

When applying Theorem 2 it is sometimes convenient to note that (2.1) and (2.2) can be simplified if \( h \) is in \( \mathcal{D}(B^{*}) \). (2.1) becomes

\[
(2.1)^{\prime} \quad Bf = B(B^{*}B + \lambda I)^{-1}B^{*}h
\]
and (2.2) becomes
\[(2.2)': \| (B^*B + \lambda I)^{-1}B^*h \| = M.\]

As an elementary application we have

**Theorem 3.** Let \( \mathcal{D}(D) \) be the set of all \( h \) in the complex Lebesgue space \( \mathcal{H} = L^2(\mathbb{R}) \) such that \( h \) is locally absolutely continuous on \( (-\infty, \infty) \) and whose derivative \( h' \) is in \( \mathcal{H} \). \( D \) sends any \( b \) in \( \mathcal{D}(D) \) into \( ib' \). Let \( h \) be in \( L^2(\mathbb{R}) \) and \( M \) be a positive real number.

Then the extremal problem \( \inf \{ \| h - Db \| : \| b \| \leq M \} \) has a unique extremal element \( Df \). Either \( h = Df \) or
\[(3.1) \quad (Df)(x) = h(x) - \frac{\omega}{2} \int_{-\infty}^{\infty} h(t) \exp(-\omega |x - t|) dt \]
for some real positive number \( \omega \).

**Proof.** As shown in Stone [4, pp. 441-445], \( D = \mathcal{F} \mathcal{F}^* \), where \( \mathcal{F} \) is the Fourier-Plancherel transform and \( \mathcal{F} \) is the self-adjoint operator that sends \( f(x) \) into \( xf(x) \). Then by Theorem 2 either \( h = Df \), or, by \( (2.1)' \), \( Df = D^2(D^2 + \omega^2 I)^{-1}h = h - \omega^2 \mathcal{F}(\mathcal{F}^2 + \omega^2 I)^{-1}\mathcal{F}^*h \), where \( \omega > 0 \). Thus (3.1) holds.

**References**