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WEAK LIMITS OF POWERS OF A CONTRACTION IN HILBERT SPACE[1]

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Let T be an operator, on the Hilbert space H, with $\|T\| \leqq 1$. Let

$$H_0 = \{x \mid \text{weak lim } T^n x = 0\}, \qquad H_1 = H_0^{\perp}.$$

We shall use the facts, proved in [1], that

1. $x \in H_0$ *if and only if* $\lim (T^n x, x) = 0$ (Theorem 3.1).
2. *On* H_1 *the operator* T *is unitary* (Theorem 1.1).

Given $x \in H$ let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. The purpose of this note is to find conditions, on the sequence $(T^n x, x)$, that will imply that x_1 is generated by eigenvectors of T. This is related to the notion of mixing for ergodic transformations.

THEOREM 1. *Let* y *be in the subspace generated by* $T^n x$, $T^{*n} x$, $n = 1, 2, \cdots$. *If* $\lim (T^n y, x) = 0$, *then* weak $\lim T^n y = 0$.

PROOF. Since H_0 and H_1 are invariant under T, it is enough to prove the theorem for the case when $x \in H_1$ (and thus also $y \in H_1$). If n_i is any subsequence of the integers and $z =$ weak $\lim T^{n_i} y$, then z is orthogonal to $T^k x$, $k = 0, \pm 1, \pm 2, \cdots$ (since T is unitary on H_1). But z belongs to the subspace generated by $T^{\pm n} x$. Hence, $z = 0$ and, thus, weak $\lim T^n y = 0$.

COROLLARY. *Let* $P(\lambda)$ *be a polynomial whose roots of modulus one are* $\lambda_1, \cdots, \lambda_k$. *If* $\lim (T^n P(T) x, x) = 0$, *then* $x = x_0 + x_1$ *where*: $x_0 \in H_0$, $x_1 = \sum_{i=1}^{k} z_i$, $T z_i = \lambda_i z_i$.

PROOF. Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. By Theorem 1,

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$P(T)x \in H_0$ and, thus, $P(T)x_1 \in H_0$ or $P(T)x_1 = 0$. The theorem now follows from the spectral representation of T on H_1.

Consider the example: $H = L_2([0, 2\pi), \mu)$ and $(Tf)(\theta) = e^{i\theta} f(\theta)$. Then

$$(T^n x, x) = \int_0^{2\pi} e^{in\theta} \left| f(\theta) \right|^2 \mu(d\theta)$$

are the Fourier coefficients of $|f|^2\mu$. Thus if

$$a_0(T^n x, x) + a_1(T^{n+1}x, x) + \cdots + a_p(T^{n+p}x, x) \xrightarrow[n\to\infty]{} 0,$$

then

$$f = f_0 + f_1, \qquad \int e^{in\theta} \left| f_0(\theta) \right|^2 \mu(d\theta) \to 0$$

and

$$f_1 = 0 \text{ except for the points } \theta_1, \cdots, \theta_k, \text{ which}$$
$$\text{are the roots of } a_0 + a_1 e^{i\theta} + \cdots + a_p e^{ip\theta} = 0.$$

Consider, for any contraction T, the polynomial $\lambda - 1$:

If $\lim((T^n x, x) - (T^{n-1}x, x)) = 0$, then $\lim(T^n x, x)$ exists. In fact, weak lim $T^n x$ exists since $x = x_0 + x_1$, where $Tx_1 = x_1$.

In [2] Sucheston proved that if T is given by an ergodic transformation on a measure space and $\lim((T^n x, x) - (T^{n-1}x, x)) = 0$ for n outside of a lacunary sequence and for every $x \in H$, then T is strongly mixing. In our treatment we cannot exclude a lacunary sequence but T can be any contraction, and the condition is assumed for one x only.

Let $g(\theta) = f(e^{i\theta})$ be a continuous function of bounded variation. Let α_k be the Fourier coefficients of g. It is known that the Fourier sums converge uniformly to g.

LEMMA A. *Let U be a unitary operator; then* $(U^n f(U)x, x) = \sum_{k=-\infty}^{\infty} (U^{n+k}x, x)\alpha_k$.

PROOF. Let $E(\theta)$ be the spectral measure of U, $0 \leqq \theta < 2\pi$. Then

$$(U^n f(U)x, x) = \int_0^{2\pi} g(\theta) e^{in\theta} (E(d\theta)x, x) = \sum_{k=-\infty}^{\infty} \alpha_k \int_0^{2\pi} e^{i(n+k)\theta} (E(d\theta)x, x).$$

LEMMA B. *The sum $\sum_{k=-\infty}^{\infty} (T^{n+k}x, x)\alpha_k$ is convergent and if $x_0 \in H_0$ then*

$$\sum_{k=-\infty}^{\infty} (T^{n+k}x_0, x_0)\alpha_k \xrightarrow[n\to\infty]{} 0.$$

PROOF. Let U be the unitary dilation of T, in the sense of [3]. Then

$$\sum_{k=-\infty}^{\infty} (T^{n+k}x, x)\alpha_k = \sum_{k=-\infty}^{\infty} (U^{n+k}x, x)\alpha_k = (U^n f(U)x, x).$$

Now if weak lim $T^n x_0 = 0$, then $(U^n x_0, x_0) = (T^n x_0, x_0) \to 0$, hence, weak lim $U^n x =$ weak lim $U^{*n} x = 0$. Thus

$$(U^n f(U)x, x) = (f(U)x, U^{*n}x) \to 0.$$

THEOREM 2. *Let g vanish on at most a countable set. If*

$$\lim_{n\to\infty} \sum_{k=-\infty}^{\infty} (T^{n+k}x, x)\alpha_k = 0,$$

then $x = x_0 + x_1$, where $x_0 \in H_0$ and x_1 is generated by eigenvectors of T.

PROOF. Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. Now

$$\sum_{k=-\infty}^{\infty} (T^{n+k}x, x)\alpha_k = \sum_{k=-\infty}^{\infty} (T^{n+k}x_0, x_0)\alpha_k + \sum_{k=-\infty}^{\infty} (T^{n+k}x_1, x_1)\alpha_k.$$

By Lemma B the first sum tends to zero; thus $\sum_{k=-\infty}^{\infty} (T^{n+k}x_1, x_1)\alpha_k \to 0$. Hence, by Lemma A and Theorem 1, $f(T)x_1 = 0$. The theorem now follows from the spectral representation of T on H_1.

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