References


University of Wisconsin and
University of Connecticut

WEAK LIMITS OF POWERS OF A CONTRACTION IN HILBERT SPACE¹

S. R. FOGUEL

Let $T$ be an operator, on the Hilbert space $H$, with $\|T\| \leq 1$. Let

$$H_0 = \{ x | \text{weak lim } T^n x = 0 \}, \quad H_1 = H_0 ^\perp.$$

We shall use the facts, proved in [1], that

1. $x \in H_0$ if and only if $\lim (T^n x, x) = 0$ (Theorem 3.1).
2. On $H_1$ the operator $T$ is unitary (Theorem 1.1).

Given $x \in H$ let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. The purpose of this note is to find conditions, on the sequence $(T^n x, x)$, that will imply that $x_1$ is generated by eigenvectors of $T$. This is related to the notion of mixing for ergodic transformations.

Theorem 1. Let $y$ be in the subspace generated by $T^n x$, $T^{n+1} x$, $n = 1, 2, \cdots$. If $\lim (T^n y, x) = 0$, then weak lim $T^n y = 0$.

Proof. Since $H_0$ and $H_1$ are invariant under $T$, it is enough to prove the theorem for the case when $x \in H_1$ (and thus also $y \in H_1$). If $n_i$ is any subsequence of the integers and $z = \text{weak lim } T^{n_i} y$, then $z$ is orthogonal to $T^k x, k = 0, \pm 1, \pm 2, \cdots$ (since $T$ is unitary on $H_1$). But $z$ belongs to the subspace generated by $T^{\pm n} x$. Hence, $z = 0$ and, thus, weak lim $T^n y = 0$.

Corollary. Let $P(\lambda)$ be a polynomial whose roots of modulus one are $\lambda_1, \cdots, \lambda_k$. If $\lim (T^n P(T) x, x) = 0$, then $x = x_0 + x_1$ where: $x_0 \in H_0$, $x_1 = \sum_{i=1}^k z_i, T z_i = \lambda_i z_i$.

Proof. Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. By Theorem 1,

¹ This work was partially supported by NSF Grants GP87 and G14736 at Northwestern University.
Consider the example: $H = L_2([0, 2\pi), \mu)$ and $(Tf)(\theta) = e^{i\theta}f(\theta)$. Then

$$
(T^n x, x) = \int_0^{2\pi} e^{in\theta} \left| f(\theta) \right|^2 \mu(d\theta)
$$

are the Fourier coefficients of $|f|^2\mu$. Thus if

$$a_0(T^n x, x) + a_1(T^{n+1} x, x) + \cdots + a_p(T^{n+p} x, x) \longrightarrow 0, \quad n \to \infty,$$

then

$$f = f_0 + f_1, \quad \int e^{in\theta} \left| f_0(\theta) \right|^2 \mu(d\theta) \to 0$$

and

$$f_1 = 0 \text{ except for the points } \theta_1, \ldots, \theta_t, \text{ which are the roots of } a_0 + a_1 e^{i\theta} + \cdots + a_p e^{ip\theta} = 0.$$

Consider, for any contraction $T$, the polynomial $\lambda - 1$:

If $\lim ((T^n x, x) - (T^{n-1} x, x)) = 0$, then $\lim (T^n x, x)$ exists. In fact, weak $T^n x$ exists since $x = x_0 + x_1$, where $Tx_1 = x_1$.

In [2] Sucheston proved that if $T$ is given by an ergodic transformation on a measure space and $\lim ((T^n x, x) - (T^{n-1} x, x)) = 0$ for $n$ outside of a lacunary sequence and for every $x \in H$, then $T$ is strongly mixing. In our treatment we cannot exclude a lacunary sequence but $T$ can be any contraction, and the condition is assumed for one $x$ only.

Let $g(\theta) = f(e^{i\theta})$ be a continuous function of bounded variation. Let $a_k$ be the Fourier coefficients of $g$. It is known that the Fourier sums converge uniformly to $g$.

**Lemma A.** Let $U$ be a unitary operator; then $(U^nf(U)x, x) = \sum_{k=0}^{\infty} (U^{n+k} x, x)a_k$.

**Proof.** Let $E(\theta)$ be the spectral measure of $U$, $0 \leq \theta < 2\pi$. Then

$$(U^nf(U)x, x) = \int_0^{2\pi} g(\theta)e^{in\theta} (E(d\theta)x, x) = \sum_{k=-\infty}^{\infty} a_k \int_0^{2\pi} e^{i(n+k)\theta} (E(d\theta)x, x).$$

**Lemma B.** The sum $\sum_{k=-\infty}^{\infty} (T^{n+k} x_0, x_0)a_k$ is convergent and if $x_0 \in H_0$ then

$$\sum_{k=-\infty}^{\infty} (T^{n+k} x_0, x_0)a_k \longrightarrow 0.'
Proof. Let $U$ be the unitary dilation of $T$, in the sense of [3]. Then
\[
\sum_{k=0}^{\infty} (T^{n+k}x, x)\alpha_k = \sum_{k=0}^{\infty} (U^{n+k}x', x)\alpha_k = (U^n f(U)x, x).
\]
Now if weak lim $T^n x_0 = 0$, then $(U^n x_0, x_0) = (T^n x_0, x_0) \to 0$, hence, weak lim $U^n x = \text{weak lim } U^{n*} x = 0$. Thus
\[
(U^n f(U)x, x) = (f(U)x, U^{n*} x) \to 0.
\]

Theorem 2. Let $g$ vanish on at most a countable set. If
\[
\lim_{n \to \infty} \sum_{k=-\infty}^{\infty} (T^{n+k} x, x)\alpha_k = 0,
\]
then $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1$ is generated by eigenvectors of $T$.

Proof. Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. Now
\[
\sum_{k=-\infty}^{\infty} (T^{n+k} x, x)\alpha_k = \sum_{k=-\infty}^{\infty} (T^{n+k} x_0, x_0)\alpha_k + \sum_{k=-\infty}^{\infty} (T^{n+k} x_1, x_1)\alpha_k.
\]
By Lemma B the first sum tends to zero; thus $\sum_{k=-\infty}^{\infty} (T^{n+k} x_1, x_1)\alpha_k \to 0$. Hence, by Lemma A and Theorem 1, $f(T)x_1 = 0$. The theorem now follows from the spectral representation of $T$ on $H_1$.

Bibliography


Northwestern University and
Hebrew University, Jerusalem, Israel