A NOTE ON SEMIGROUPS OF OPERATORS 
ON A LOCALLY CONVEX SPACE

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1. The object of this note is to generalize certain results in the theory of semigroups of continuous linear operators on a Banach space to the case where, instead of a Banach space, we consider a locally convex space. A family \( \{T(\xi)\}_{\xi \geq 0} \) of linear operators on a vector space is called a semigroup if \( T(\xi + \eta) = T(\xi) \circ T(\eta) \), \( \xi, \eta \in (0, \infty) \). E. Hille [4] and N. Dunford [2] have proved that if \( \{T(\xi)\}_{\xi \geq 0} \) is a semigroup of bounded linear operators on a Banach space \( E \) such that for every \( x \in E \), \( \xi \rightarrow T(\xi)x \) is a measurable function from \( (0, \infty) \) into \( E \) and such that \( \|T(\xi)\|_{E \rightarrow E} \leq 1 \) is bounded for every \( \delta > 0 (\delta < 1) \), then \( \xi \rightarrow T(\xi)x \) is a continuous function from \( (0, \infty) \) into \( E \) for \( x \in E \). Proposition 2 is an analogue of this result while Propositions 3 and 4 are analogues of results due to R. S. Phillips [5] and P. Lax, respectively.

2. Definition 1. Let \( E \) be a locally convex space, \( S \) a set and \( \mathcal{M} \) a \( \sigma \)-ring of “measurable” subsets of \( S \). A function \( x: S \rightarrow E \) \((\xi \rightarrow x(\xi))\) is called measurable if it is the limit, almost everywhere, of a sequence of countably valued functions. (A function is said to be countably valued if its range is countable and it takes each value different from zero on a measurable set.)

Remark. If \( x(\xi) \) is a measurable \( E \)-valued function and \( p \) is any continuous seminorm on \( E \), then \( p(x(\xi)) \) is a real-valued measurable function on \( S \).

Proposition 1. Let \( S = (0, \infty) \) with \( \mathcal{M} \) the \( \sigma \)-ring of Lebesgue measurable subsets of \( (0, \infty) \) and \( x(\sigma) \) be a measurable function from \( (0, \infty) \) into \( E \). Then for any continuous seminorm \( q \) on \( E \), there exists a sequence \( \{x_n(\sigma)\}_{n=1,2,...} \) of countably valued functions such that \( q(x(\sigma) - x_n(\sigma)) \rightarrow 0 \) as \( n \rightarrow \infty \) uniformly for \( \sigma \) outside a set of measure zero.

We shall prove the proposition under a weaker hypothesis, viz., that \( x(\sigma) \) is weakly measurable (i.e., for every continuous linear functional \( x' \) on \( E \), \( \langle x(\sigma), x' \rangle \) is a measurable function of \( \sigma \) on \( (0, \infty) \)) and that \( x(\sigma) \) is almost separably valued (i.e., \( x(\sigma) \) belongs to a separable subspace of \( E \) almost everywhere).

We may suppose that the range of \( x(\sigma) \) is contained in a separable subspace \( L \) of \( E \) and find a sequence \( \{x_n(\sigma)\}_{n \geq 1} \) of countably valued...
functions such that \( g(x_n(\sigma) - x(\sigma)) \) tends to zero as \( n \) tends to infinity uniformly for \( \sigma \) in \((0, \infty)\). Let \( \mathcal{L} \) denote the quotient space of \( \mathcal{L} \), on which \( g \) defines a norm \( \tilde{g} \). The completion \( \tilde{\mathcal{L}}^\sim \) of \( \mathcal{L} \) is a separable Banach space. Hence its conjugate space \((\tilde{\mathcal{L}}^\sim)'\) contains a sequence \( \{y_n^*\} \) such that

\[
\tilde{g}(\tilde{x}) = \sup_n |\langle \tilde{x}, y_n^* \rangle| \quad \text{for } \tilde{x} \in \tilde{\mathcal{L}}^\sim.
\]

Now \( x \to y_n^*(x) \), where \( \tilde{x} \) is the image of \( x \) by the quotient map \( L \to \tilde{L} \), is a continuous linear functional on \( L \). By the Hanh-Banach theorem, there exists a continuous linear functional \( x^* \) on \( E \) such that

\[
\langle \tilde{x}, y_n^* \rangle = \langle x, x_n^* \rangle \quad \text{for } x \in L \text{ and } n = 1, 2, \ldots.
\]

Now \( q(x(\sigma)) = \tilde{q}(\tilde{x}(\sigma)) = \sup_n |\langle x(\sigma), x_n^* \rangle| \) is measurable as \( \langle x(\sigma), x_n^* \rangle \) is measurable for \( n = 1, 2, \ldots \). Similarly, for any \( a \in L \), \( q(x(\sigma) - a) \) is a measurable function. Let \( A = \{ \sigma | q(x(\sigma)) > 0 \} \). If \( \{a_i\} \) is a sequence dense in \( L \) and \( A = A \cap \{ a_i q(x(\sigma) - a_i) \} < 1/n \) then \( A = \bigcup a_i A_i \). Let \( B_i = A_i, B_j = A_i - \bigcup_{i=1}^{j-1} B_i \) \((j > 1)\). The \( B_i \) are disjoint measurable and \( \bigcup B_i = \bigcup A_i = A \).

Let

\[
x_n(\sigma) = a_i \quad \text{for } \sigma \in B_i,
\]

\[
= 0 \quad \text{for } \sigma \notin A.
\]

Then \( q(x(\sigma) - x_n(\sigma)) < 1/n \) for all \( \sigma \) so that \( q(x(\sigma) - x_n(\sigma)) \to 0 \) as \( n \to \infty \) uniformly for \( \sigma \) in \((0, \infty)\).

**Definition 2.** A semigroup of operators on a locally convex space \( E \) is said to be measurable if, for \( x \in E \), the function \( \xi \to T(\xi)x \) is measurable from \((0, \infty)\) to \( E \).

**Proposition 2.** If \( \{ T(\xi) \}_{\xi > 0} \) is a measurable semigroup of continuous linear operators in a locally convex space \( E \) such that, for every \([\alpha, \beta] \subset (0, \infty)\), \( \{ T(\tau) \}_{\alpha < \tau < \beta} \) is an equicontinuous family of operators, then \( \xi \to T(\xi)x \) is continuous for every \( x \in E \).

**Proof.** We have to show that for \( \xi \in (0, \infty) \) and \( x \in E \),

\[
(1) \quad T(\xi \pm \eta)x - T(\xi)x \to 0 \quad \text{in } E \text{ as } \eta \to 0.
\]

Let \( 0 < \alpha < \beta < \xi \). As \( \{ T(\tau) \}_{\tau \in [\alpha, \beta]} \) is an equicontinuous set, given any continuous seminorm \( p \) on \( E \), there exists a continuous seminorm \( g \) on \( E \) and \( k > 0 \) such that

\[
(2) \quad p(T(\tau)y) \leq kg(y) \quad \text{for } \alpha \leq \tau \leq \beta \text{ and } y \in E.
\]

Let \( \eta > 0 \) be such that \( \alpha - \eta > 0 \) and \( \beta < \xi - \eta \) for \( 0 < \eta < \eta_0 \). By (2), we have
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\[ p(T(\xi \pm \eta) - T(\xi)x) = p(T(\tau)[T(\xi \pm \eta - \tau)x - T(\xi - \tau)x]) \leq kq(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x). \]

As \( \{ T(\xi \pm \eta - \tau) \}_{\tau \in [\alpha, \beta]} \) and \( \{ T(\xi - \tau) \}_{\tau \in [\alpha, \beta]} \) are equicontinuous sets, \( \{ T(\xi \pm \eta - \tau)x \}_{\tau \in [\alpha, \beta]} \) and \( \{ T(\xi - \tau)x \}_{\tau \in [\alpha, \beta]} \) are bounded subsets of \( E \) so that \( q(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x) \) is a bounded measurable function of \( \tau \) in \( [\alpha, \beta] \). Integrating (3) with respect to \( \tau \) from \( \alpha \) to \( \beta \) we get

\[ (\beta - \alpha)p(T(\xi \pm \eta)x - T(\xi)x) \leq k \int_{\alpha}^{\beta} q(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x) \, d\tau. \]

The integral on the right-hand side tends to zero with \( \eta \) if we show that, for given \( \epsilon > 0 \), there exists a continuous \( E \)-valued function \( f_*(\tau) \) such that

\[ \int_{\alpha - \epsilon_0}^{\beta + \epsilon_0} q(T(\xi - \tau)x - f_*(\tau)) \, d\tau < \epsilon. \]

For then

\[
\begin{align*}
\int_{\alpha}^{\beta} q(T(\xi \pm \eta - \tau)x - T(\xi - \tau)x) \, d\tau & \leq \int_{\alpha}^{\beta} q(T(\xi \pm \eta - \tau)x - f_*(\tau \mp \eta)) \, d\tau + \int_{\alpha}^{\beta} q(f_*(\tau \mp \eta) - f_*(\tau)) \, d\tau \\
& + \int_{\alpha}^{\beta} q(f_*(\tau) - T(\xi - \tau)x) \, d\tau
\end{align*}
\]

and the first and the third integrals are each majorised by \( \epsilon \) and the second integral tends to zero with \( \eta \) since \( f_*(\tau) \) is continuous. Now \( h(\tau) = T(\xi - \tau)x \) being measurable from proposition (1), it follows that, given \( \epsilon > 0 \), there exists a countably valued function \( x(\tau) \) such that \( \int_{\alpha - \eta_0}^{\beta + \eta_0} q(h(\tau) - x(\tau)) \, d\tau < \epsilon/3 \). Let \( x(\tau) = \sum_{i=1}^{n} a_i \chi_{A_i} \) where \( a_i \in E \) and \( \chi_{A_i} \) are characteristic functions of disjoint measurable sets \( A_i \) contained in \( [\alpha - \eta_0, \beta + \eta_0] \). Then we can choose \( m \) such that for \( y(\tau) = \sum_{i=1}^{n} a_i \chi_{A_i}, \int_{\alpha - \eta_0}^{\beta + \eta_0} q(x(\tau) - y(\tau)) \, d\tau < \epsilon/3 \). Let \( q(y(\tau)) \leq M. \) Given \( \delta > 0 \) we can find compact sets \( K_i \subset A_i \) and open sets \( O_i \supset A_i \) \( (i = 1, 2, \ldots, m) \) such that \( \sum_{i=1}^{m} m(O_i - K_i) < \delta \) \( (m \) being Lebesgue measure). For every \( i \) there exists a continuous function \( g_i(\tau), 0 \leq g_i(\tau) \leq 1, \) such that \( g_i(\tau) \) equals 1 on \( K_i \) and 0 outside \( O_i \). If \( g(\tau) = \sum_{i=1}^{n} a_i g_i(\tau), \) then \( q(g(\tau)) < mM \) and \( g(\tau) \) equals \( y(\tau) \) for \( \tau \in \bigcup_{i=1}^{m} (O_i - K_i) \) so that \( \int_{\alpha - \eta_0}^{\beta + \eta_0} q(y(\tau) - g(\tau)) \, d\tau < 2mM \delta < \epsilon/3 \) for \( \delta < \epsilon/6mM. \) Taking \( f_*(\tau) = g(\tau), \) (5) is satisfied.
Remark 1. Proposition 2 remains true if instead of the hypothesis that $\xi \mapsto T(\xi)x$ are $E$-valued measurable functions for $x \in E$, we have the weaker hypothesis that these functions are weakly measurable and almost separably valued. For, from the proof of Proposition 1, given $\epsilon > 0$, there exists a countably valued function $x(\tau)$ such that $\int_{\mathbb{R}^+} \frac{1}{q} \left| h(\tau)x - x(\tau) \right| d\tau < \epsilon/3$ and the proof is completed by the same method as above.

Remark 2. The converse of Proposition 2, viz., the statement that "If $\{ F(\xi) \}_{\xi > 0}$ is a semigroup such that $\xi \mapsto F(\xi)x$ are continuous functions for $x \in E$, then $\{ F(\xi) \}_{\xi > 0}$ is a measurable semigroup such that for every $[a, \beta] \subset (0, \infty)$, $\{ F(\xi) \}_{\xi \in [a, \beta]}$ is an equicontinuous set of maps of $E$ into $E$" is true if $E$ is barrelled (tonnelé) or is the strong dual of a metrisable locally convex space. For, if $\xi \mapsto F(\xi)x$ is continuous, then it is measurable and maps the compact set $[a, \beta] \subset (0, \infty)$ onto a compact subset of $E$. Thus $\{ F(\xi) \}_{\xi \in [a, \beta]}$ is compact in $\mathcal{L}(E, E)$, the space of linear continuous maps of $E$ into $E$ furnished with the topology of simple convergence, and therefore is a closed bounded subset of $\mathcal{L}(E, E)$. If $E$ is barrelled, this implies that $\{ F(\xi) \}_{\xi \in [a, \beta]}$ is equicontinuous. If $E$ is the strong dual of a metrisable space, let $(\xi_n)_{n=1, 2, ...}$ denote the sequence of rationals in $[a, \beta]$. Then $\{ T(\xi_n) \}_{n=1, 2, ...}$ being a subset of $\{ T(\xi) \}_{\xi \in [a, \beta]}$ and is therefore equicontinuous [3, Proposition 1, p. 62]. Hence its closure in $\mathcal{L}(E, E)$ is equicontinuous. Now, for any $\sigma \in [a, \beta]$, there exists a subsequence $(\xi_{n_k})$ of $(\xi_n)$ such that $\xi_{n_k} \to \sigma$ as $k \to \infty$, so that $T(\sigma)x = \lim T(\xi_{n_k})x$ for every $x \in E$. Thus $\{ T(\xi) \}_{\xi \in [a, \beta]}$ is the closure of $\{ T(\xi_n) \}_{n=1, 2, ...}$ in $\mathcal{L}(E, E)$ which is equicontinuous.

Proposition 3. Let $\{ T(\xi) \}_{\xi > 0}$ be a measurable semigroup of continuous linear operators in a Fréchet space $E$. Then for any $[a, \beta] \subset (0, \infty)$,

$$\{ T(\xi) \}_{\xi \in [a, \beta]}$$

is an equicontinuous set of maps of $E$ into $E$.

Proof. Let $\{ p_n \}$ be a sequence or seminorms on $E$ defining the topology of $E$. Since $E$ is Fréchet, it is sufficient to prove that $\{ T(\xi)x \}_{\xi \in [a, \beta]}$ is bounded in $E$ for any fixed $x \in E$. If this is not true, there exists an $x \in E$ and an integer $i_0$ and a sequence $\{ \xi_n \}_{n=1, 2, ...}$ tending to a real number $\gamma$ where $\xi_n, \gamma \in [a, \beta]$, such that

$$p_n(T(\xi_n)x) \geq n \quad \text{for all } n.$$

Let $\{ F_i \}_{i=1, 2, ...}$ be a sequence of measurable subsets of $(0, \gamma)$ such that
(i) $F_{i+1} \subseteq F_i$,
(ii) the measure $m(F_i) > \gamma/2 + \gamma/3i$,
(iii) the measurable function $p_i(T(\xi)x)$ is bounded on $F_i$ by $M_i$, say.

Clearly such a sequence exists and $F = \bigcap_{i=1}^\infty F_i$ is measurable with $m(F) \geq \gamma/2$ and $p_i\{T(\xi)x\} \leq M_i$ for $\xi \in F$, and $i = 1, 2, \ldots$, i.e., $\{T(\xi)x\}_{\xi \in F}$ is a bounded subset of $E$.

The sets

$$A_n = \{\xi_n - \eta \mid \eta \in F \cap (0, \xi_n)\}, \quad n = 1, 2, \ldots$$

are measurable and

(1) \quad $m(A_n) \geq \frac{\gamma}{4}$ \quad for $n \geq N$, say.

For $\eta \in F \cap (0, \xi_n)$,

$$n = p_{i_0}[T(\xi_n)x] \leq p_{i_0}[T(\xi_n - \eta)T(\eta)x].$$

Let $\sigma \in A_n$ be arbitrary. Then

$$\sigma = \xi_n - \eta \quad \text{for some } \eta \in F \cap (0, \xi_n),$$

so that

(2) \quad $p_{i_0}[T(\sigma)T(\eta)x] \geq n$.

Let $A = \limsup_{n \to \infty} A_n$. Then $m(A) \geq \gamma/4$ by (1). For $\sigma_0 \in A$, $p_{i_0}$ is unbounded on $T(\sigma_0)\{T(\eta)x\}_{\eta \in F}$ by (2), i.e., $T(\sigma_0)\{T(\eta)x\}_{\eta \in F}$ is not a bounded subset of $E$. This is a contradiction since $T(\sigma_0)$ is a continuous linear map of $E$ into $E$ and $\{T(\eta)x\}_{\eta \in F}$ is a bounded set in $E$.

Combining Propositions 2 and 3 we have the

THEOREM. Every measurable semigroup of continuous linear operators on a Fréchet space is a continuous function from $(0, \infty)$ into the space of continuous linear maps of $E$ into $E$ furnished with the topology of simple convergence.

3. PROPOSITION 4. Let $\{T(\xi)\}_{\xi \geq 0}$ be a measurable semigroup of continuous linear operators on a locally convex space $E$ such that $\{T(\xi)\}_{\alpha \leq \xi \leq \beta}$ is an equicontinuous family for each $[\alpha, \beta] \subset (0, \infty)$ and such that for some $\delta > 0$, $T(\xi_0)$ is a compact operator on $E$. Then, for any $\alpha > \xi_0$, $T(\alpha)$ is a compact operator and $\xi \to T(\xi)$ is a continuous mapping of $(\alpha, \infty)$ into $\mathcal{L}_{\mathcal{B}}(E, E)$ where $\mathcal{L}_{\mathcal{B}}(E, E)$ is the space of continuous linear maps of $E$ into $E$ furnished with the topology of uniform convergence on bounded subsets of $E$.
Proof. Let $b > a > \xi_0$ and let $B$ be a bounded subset of $E$, $V$ a neighbourhood of 0 in $E$ and

$$W(B, V) = \{ u \in \text{L}(E, E) \mid u(B) \subset V \}.$$  

We shall show that there exists $\delta > 0$ such that

$$T(\xi + \eta) - T(\xi) \in W(B, V)$$

for $|\eta| < \delta$, $a < \xi < b$, $a < \xi + \eta < b$.

As the sets $\{ W(B, V) \}$, where $B$ is a bounded subset of $E$ and $V$ is a neighbourhood of 0 in $E$, form a fundamental system of neighbourhoods of zero in $\text{L}(E, E)$, this will prove that $T(\xi + \eta) - T(\xi) \to 0$ in $\text{L}(E, E)$ as $\eta \to 0$. Let $V_1$ be a neighbourhood of 0 in $E$ such that

(1) \hspace{1cm} V_1 + V_1 + V_1 \subset V.

Let $V_2$ be a neighbourhood of 0 in $E$ such that

(2) \hspace{1cm} V_2 \subset V_1

and

(3) \hspace{1cm} T(\xi - \xi_0) V_2 \subset V_1 \quad \text{for} \hspace{0.5cm} a \leq \xi \leq b.

This is possible since $\{ T(\xi - \xi_0) \}_{\xi \in [a, b]}$ is an equicontinuous family of maps of $E$ into $E$.

Now, $T(\xi_0)$ being a compact operator, there exists a neighbourhood $U$ of 0 such that $T(\xi_0) U$ is a relatively compact subset of $E$. The bounded subset $B$ of $E$ is absorbed by $U$, so that $T(\xi_0) B$ is also relatively compact. Given a neighbourhood $V_2$ of 0 in $E$, there exist $x_1, x_2, \cdots, x_n \in E$ such that

$$T(\xi_0) B \subset \bigcup_{k=1}^{n} \{ T(\xi_0) x_k + V_2 \},$$

i.e., for any $x \in B$, there exists an $x_k$ such that

(4) \hspace{1cm} T(\xi_0) x - T(\xi_0) x_k \in V_2.

By Proposition 1, for any $x \in E$, $\xi \to T(\xi) x$ is a continuous function from $(0, \infty)$ into $E$. In particular, $\xi \to T(\xi) x_k$ are continuous functions for $k = 1, 2, \cdots, n$. We can, therefore, find $\delta > 0$ such that, for $k = 1, 2, \cdots, n$,

$$T(\xi + \eta) x_k - T(\xi) x_k \in V_2 \quad \text{if} \quad |\eta| < \delta, \ a \leq \xi, \ \xi + \eta \leq b.$$  

Let $x \in B$ and $x_k$ be as in (4). Then
\begin{align*}
T(\xi + \eta)x - T(\xi)x &= T(\xi + \eta)x - T(\xi + \eta)x_k + T(\xi + \eta)x_k - T(\xi)x_k \\
&+ T(\xi)x_k - T(\xi)x \\
&= T(\xi + \eta - \xi_0)[T(\xi_0)x - T(\xi_0)x_k] \\
&+ [T(\xi + \eta)x_k - T(\xi)x_k] \\
&+ T(\xi - \xi_0)[T(\xi_0)x_k - T(\xi_0)x] \\
&\subseteq T(\xi + \eta - \xi_0)V_2 + V_2 + T(\xi - \xi_0)V_2 \\
&\subseteq V_1 + V_1 + V_1 \subseteq V;
\end{align*}

i.e., \([T(\xi + \eta) - T(\xi)]B \subseteq V\) for \(|\eta| < \delta, a \leq \xi + \eta \leq b, \xi_0 < a\). This proves the required continuity of the function \(\xi \mapsto T(\xi)\).

That \(T(\xi)\) is compact for \(\xi > \xi_0\) follows from the fact \(T(\xi_0)\) is a compact operator and \(T(\xi) = T(\xi - \xi_0)T(\xi_0)\) where \(T(\xi - \xi_0)\) is a continuous linear map of \(E\) into \(E\).

\section*{References}


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\section*{ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS$^1$}

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\section{Introduction.} Consider a set \(\{P_n(x)\}\) of orthogonal polynomials defined by the classical recurrence formula,

\begin{equation}
\begin{aligned}
P_n(x) &= (x - c_n)P_{n-1}(x) - \lambda_nP_{n-2}(x) \quad (n = 1, 2, 3, \ldots), \\
P_{-1}(x) &= 0, \quad P_0(x) = 1, \quad c_n \text{ real}, \quad \lambda_{n+1} > 0.
\end{aligned}
\end{equation}

In [2], the author initiated a study of (1.1) based on the chain sequences of Wall [6], the fundamental relation being that the zeros

\footnotesize
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