DIFFERENTIAL OPERATORS APPROXIMATELY OF THE
FORM \((1-(1/W(x))d/dx)^n\)

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1. Introduction. This paper studies the factorization of certain
homogeneous linear differential operators \(L = \sum_{i=0}^n a_i(x)D^i\) into prod-
ucts \(b_0(x)(1-b_1(x)D) \cdots (1-b_n(x)D)\) of first-order factors, the func-
tions \(a_i\) and \(b_i\) discussed in terms of asymptotic behavior as \(x\) tends
to infinity in the complex plane. Among other results, we obtain in-
formation about the asymptotic behavior of the solutions of \(L(y) = 0\).
Speaking loosely, the operators \(L\) are assumed to be expressible in the form
\(L = (1 - W^{-1}D)^n + E_n(1 - W^{-1}D)^{n-1} + \cdots + E_1(1 - W^{-1}D) + E_0\) where each \(E_i \to 0\) as \(x \to \infty\), while \(\int E_0 W \to \infty\) as
\(x \to \infty\). In the factorization \(L = b_0(1-b_1D) \cdots (1-b_nD)\) we obtain,
it will be the case that \(b_0 \to 1\) and \(b_i/(W^{-1}) \to 1\) (i > 0) as \(x \to \infty\).

We use Strodt's asymptotic relations \(<, \sim, \approx\), referring to a sort
of sectorial neighborhood system of infinity in the complex plane.
This system is denoted \(F(\alpha, \beta)\). (For an index of notation and termi-
nology, see [2, Part IX, pp. 105-107].) Roughly, \(f < 1\) means \(f \to 0\);
\(f \sim g\) means \(f/g \to 0\); \(f \sim g\) means \(f/g \to 1\), or \((f/g) - 1 < 1\); and \(f \sim g\) means \(f/g \to c\), where \(c\) is a nonzero complex constant. Logarithmic
monomials are functions of the form \(c x^{-m_1}(\log x)^{m_2} \cdots (\log x)^{m_i}\),
with \(c\) complex and \(m_i\) real, \(\log\) being the \(i\)-fold iteration of \(\log\).
Such functions are \(< 1\) and \(> 1\) according as the first nonzero
\(m_i\) is negative or positive. A logarithmic domain is a set of functions
which, together with all finite linear combinations thereof (with
logarithmic monomials for coefficients), are either \(\sim\) logarithmic
monomials or else are trivial (\(\sim\) every logarithmic monomial). The
restriction, in Theorem I, to a logarithmic domain \(\mathcal{D}\) has the effect of
guaranteeing that the differences \(V - W\), where \(V \sim W\), will be \(\leq\)-com-
parable with \(W_0/W_0\) (which is \(\approx x^{-1}(\log x)^{-1} \cdots (\log x)^{-1}\) for
some \(q\)).

The letters \(W, W_i, V, U\), etc. are used for functions which are \(\sim\)
logarithmic monomials of the form
\[c x^{-1}(\log x)^{-1} \cdots (\log^{p-1} x)^{-1}(\log p x)^{-1+\tau}
\cdot (\log^{p+1} x)^{a_1} \cdots (\log^{p+\tau} x)^{a_{\tau}}, \quad \tau > 0,\]

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so that their \( \int_{0}^{\infty} f \) diverges. (In fact, \( \int_{0}^{\infty} f \sim r^{-1}x (\log x) \cdots (\log^q x) f(x) \) for any such \( f \). See [3, Part V, Lemma \( \xi \]).) Logarithmic monomials of this sort form the divergence class. For every \( M \) in this class and every \( E<1 \), \( (M^{-1} D) E<1 \); in particular, \( D E< \alpha (\log x)^{-1} \cdots (\log^q x)^{-1} \) for \( q=0, 1, 2, \ldots \). This fact is used in Lemma 9.

An operator \( L \) is unimajoral if \( L(y)<1 \) whenever \( y<1 \), while \( L(y)\sim 1 \) whenever \( y \sim 1 \); the \( L \)'s we study are unimajoral. It is known [2, §27] that whenever the coefficients \( a_i(x) \) of a unimajoral \( L= \sum_{i=0}^{n} a_i(x) D^i \) belong to a logarithmic domain and \( a_n(x) \) is non-trivial, \( L \) can be approximately factored: an approximate factorization sequence \( W_1 \lesssim W_2 \lesssim \cdots \lesssim W_n \) of logarithmic monomials in the divergence class can be found such that \( L= W_n \cdots W_1 + \sum_{i=1}^{n} E_i W_i \) \( \cdots W_1 + E_0 \), where each \( E_i<1 \) and by \( W_i \) we mean the (unimajoral) operator \( (1-W_i^{-1} D) \). A similar expression, with different \( E_i \)'s (but always \( <1 \)), can be written using any sequence of functions \( (V_1, V_2, \ldots, V_n) \) with \( V_i \sim W_i \) in place of \( (W_1, W_2, \ldots, W_n) \). We are interested in sequences \( (V_1, V_2, \ldots, V_n) \) for which the corresponding \( E_i \) are all 0 for \( i<n \); such sequences yield exact factorizations: \( L= (1+E_n) V_n V_{n-1} \cdots V_1 \). In [3, Part II], exact factorization is achieved for the case in which \( (W_1, W_2, \ldots, W_n) \) is separated: \( \exp((W_i-W_j))<1 \) or \( >1 \) for all pairs \( i \neq j \). (For a precise definition, see [3, Part II].) This condition permits the determination of an exact factorization sequence \( (V_1, V_2, \ldots, V_n) \) from certain quasi-linear algebraic differential equations treated in [2].

The present study treats unimajoral operators which possess approximate factorization sequences \( W_1=W_2=\cdots=W_n \): \( L= (\hat{W})^n + \sum_{i=0}^{n} E_i(\hat{W})^i \). The results of [3, Part II], which at first appear useless in case of such extreme lack of separation, are applied to an operator with a separated factorization sequence obtained from \( L \) by some changes of variable. Theorem I, below, takes care of operators \( L= (\hat{W})^n + \sum_{i=0}^{n} E_i(\hat{W})^i \) whose coefficients (which are in any case \( <1 \)) satisfy \( E_i<((\hat{W})^{-n} \). We call such operators regular, this designation being suggested by analogy to the criterion for regular singular points in the classical treatment of linear differential equations.

Throughout this paper, integral signs should be construed as having a fixed lower limit, the upper limit serving as the independent variable. In situations involving \( 1/\hat{W} \), the domain of that variable will be subject to obvious and harmless restrictions.

2. Heuristics. Introduction of the new variable \( s=-\int \hat{W} \) leads to \( \hat{W}=1+d/ds \), so the operator \( L= (\hat{W})^n + \sum_{i=0}^{n} E_i(\hat{W})^i \) becomes \( (1+d/ds)^n + \sum_{i=0}^{n} E_i(1+d/ds)^i \). It is natural then to set \( y=e^{-z} \) in
the equation $L(y) = 0$; this leads to $((d/ds)^n + \sum_{i=0}^n E_i(d/ds)^i)z = 0$.

In the trivial case where each $E_i = 0$, the substitutions $s = e^t$ and $z(s) = e^{-t}v(t)$ convert the equation $(d/ds)^n z = 0$ into

$$e^{-(n+1)t}(-1)^n(1-n^{-1}d/dt)(1-(n-1)^{-1}d/dt)\cdots(1-1^{-1}d/dt)v = 0,$$

in which we have an exactly factored operator with the separated factorization sequence $(1, 2, \cdots, n)$. The solutions $v(t)$ are, of course, $\approx \exp f_1, \exp f_2, \cdots, \exp f_n$. In case the $E_i$ are not all zero, the same substitution in the equation $((d/ds)^n + \sum_{i=0}^n E_i(d/ds)^i)z = 0$ leads (by straightforward calculation) to a unimajoral equation for which $(1, 2, \cdots, n)$ is an approximate factorization sequence, provided the $E_i$ satisfy the condition $E_i e^{(n-i)t} < 1$ $(i = 0, 1, \cdots, n)$. Since $(1, 2, \cdots, n)$ is separated, we may apply Theorem II of [3] to obtain an exact factorization $V_n V_{n-1} \cdots V_1 v(t) = 0$ where $V_1 \sim 1$, $V_2 \sim 2, \cdots, V_n \sim n$, and solutions $v(t)$ which are $\approx \exp \int V_i$, $i = 1, 2, \cdots, n$.

3. **Definition.** If $L = (W)^n + \sum_{i=0}^n E_i(W)^i$ with $E_i < (f(W))^{-i}$ for $i = 0, 1, \cdots, n$, we shall say $L$ is regular with respect to $W$.

4. **Theorem I.** Let $\mathcal{D}$ be a logarithmic domain. Let $L$ be a unimajoral operator expressible in the form $(W_0)^n + \sum_{i=0}^n E_i(W_0)^i$ with each $E_i < 1$ and $W_0$ in the divergence class. Then either there exists no $W$ in $\mathcal{D}$ such that $L$ is regular with respect to $W$, or else $\{ W : W \in \mathcal{D}$ and $L$ is regular with respect to $W \}$ is precisely one of the equivalence classes in $\mathcal{D} \cap \{ W : W \sim W_0 \}$ defined by the equivalence relation $\equiv$ where $U \equiv V$ means $U - V \approx f(W_0)$. If $L$ is regular with respect to $W$, then $L$ can be expressed as a product of first-order factors: $L = uV_n \cdots V_1$, where $V_i \equiv W$ for each $i$. Conversely, if $L$ is expressible as such a product, then $L$ is regular with respect to $W$. If $L$ is regular with respect to $W$, the solutions of $L(y) = 0$ are generated by a basis consisting of functions $y_0, y_1, \cdots, y_{n-1}$ where $y_0 \approx y_1 \approx \cdots \approx y_{n-1}$. For each $i$, $y_i \approx \exp \int W_i$ for a certain $W_i \sim W$, and $\log(y_i/y_{i+1}) \approx -\log \int W$. 

**Proof.** The theorem summarizes the following lemmas.

5. **Lemma.** Let $L$ be regular with respect to $W$. Then there exist functions $W_0, W_1, \cdots, W_{n-1}$ satisfying $W_i - W \approx W/f(W)$ for $i > 0$, $W_0 - W \approx W/f(W)$, such that the equation $L(y) = 0$ has solutions $y_0, y_1, \cdots, y_{n-1}$ satisfying $y_i \approx \exp \int W_i$, $i = 0, 1, \cdots, n-1$.

**Proof.** Let $f = s^e e^{-s}$, where $s(x) = -f^2 W$. We have

$$\tilde{W}(f) = n s^n s^{-1} e^{-s}, (W)^2(f) = n (n-1) s^{n-2} e^{-s}, \cdots, (W)^n(f) = n! e^{-s}.$$
Using the relation \( \tilde{W}(Z) = (\tilde{W}(Z)) (\tilde{M}(z)) \), where \( M = WZ^{-1}\tilde{W}(Z) \), we obtain

\[
W(fz) = ns^{n-1}e^{-s}Q_s(z), (W)^2(fz) = n(n-1)s^{n-2}e^{-s}Q_2Q_s(z), \ldots, (W)^n(fz) = n!e^{-s}Q_s \ldots Q_1(z)
\]

where \( Q_i = -(n-i+1)W^i/W \) for each \( i \). Hence \( L(fz) \) may be written \( n!e^{-s}L'(z) \) where \( L' = \tilde{Q}_n \cdots \tilde{Q}_1 + \sum_{i=0}^{n-1} (E_i s^{n-i}/(n-i)!\tilde{Q}_i \cdots \tilde{Q}_1) \).

(All this is verified in an elementary way.) Since \( L \) is regular with respect to \( W \), all but the leading coefficient in \( L' \) are \( <1 \). \( L' \) is thus a unimajoral operator; furthermore, its factorization sequence \( (Q_1, \ldots, Q_n) \) is separated. It follows from [3, Theorem I] that there exists a representation of \( L' \) in the form \( vU_n \cdots U_1 \) with \( v \sim 1 \) and \( U_i \sim Q_i \) for each \( i \). According to [3, Theorem II], the equation \( L'(z) = 0 \) has solutions \( z_i \), where \( z_i \sim \exp \int U_j, j = 1, \ldots, n \). The corresponding solutions of \( L(y) = 0 \) are of the form \( e^{-st}z_i \), which may be expressed as the product of a function \( =1 \) and \( \exp \int (W + U_j + nW'/W) \). The parenthesized term serves as the \( W_i \) called for in this lemma, taking \( i = j-1 \), \( y_i = c_i e^{-st}z_{i+1} \), \( c_i \) a normalizing constant.

6. **Lemma.** In Lemma 5 we have \( y_0 < y_1 < \cdots < y_{n-1} \).

**Proof.** It is clear that \( y_i/y_{i+1} \sim \exp \int (W_i - W_{i+1}) \) and that \( W_i - W_{i+1} \sim -W'/W \). Using [3, Part V, Lemma \( \xi \)], we see that the latter is \( \sim -\tau x^{-1}(\log x)^{-1} \cdots (\log px)^{-1} \), for a positive real \( \tau \) and an integer \( p \geq 0 \). Our conclusion then follows from [3, Part V, Lemma \( \gamma \)].

7. **Lemma.** In Lemma 5 we have \( \log(y_i/y_{i+1}) \sim -\log fW \).

**Proof.** The relation \( y_i/y_{i+1} \sim \exp \int (W_i - W_{i+1}) \) immediately implies

\[
y_i/y_{i+1} = (1 + E) \exp \int (W_i - W_{i+1})
\]

where \( E < 1 \). Thus

\[
\log(y_i/y_{i+1}) = \int (W_i - W_{i+1}) + \log(1 + E).
\]

Since \( W_i - W_{i+1} \sim -W'/W, \int (W_i - W_{i+1}) \sim -\log fW \) (this is seen with the help of [3, Part V, Lemma \( \xi \)]), which is \( >1 \). The term \( \log(1 + E) \) is seen to be \( <1 \), using [1, §27]. Hence we have \( \log(y_i/y_{i+1}) \sim -\log fW \).

8. **Lemma.** If \( L \) is regular with respect to \( W \), there exist functions \( V_1, V_2, \ldots, V_n \) such that \( V_1 \sim V_2 \sim \cdots \sim V_n \sim W \), \( V_i - W \sim W'/W \) for each \( i \), and \( L = uV_n \cdots V_1 \) with \( u \sim 1 \).
Proof. As in the proof of Lemma 5, we have \( L(y) = n! e^{-v} \dot{U}_n \cdots \dot{U}_1(y/f) \). Writing \( U_1 = -n(1 + h_1)W \) with \( h_1 \ll 1 \), we have \( \dot{U}_1(y/f) = U_1(\dot{U}_1(y/f))f \) and

\[
\dot{U}_1(y/f) = U_1(s^{-n}e^*)
\]

\[
= s^{-n}e^* + s(n(1 + h_1)Ds)^{-1}(-ns^{-n-1}e^*Ds + s^{-n}e^*Ds)
\]

\[
= f^{-1}(1 + h_1)^{-1}(h_1 + s/n).
\]

Thus \( V_1 = (-nDs/s)(1 + h_1)^{-1}(1 + h_1)^{-1}((1 + h_1 + s/n)f = -Ds(h_1n/s + 1) = W(1 - h_2n/fW). \) From this we see that \( V_1 - W < W/fW \).

Now \( \dot{U}_1(y/f) = Ws^{-n}e^* \) where \( w \sim 1/n \). Operating on \( (\dot{U}_1(y/f))V_1 \) with \( \dot{U}_2 \), we get \( (\dot{U}_2(w))V_2 = U_2(s^{-n}e^* \dot{V}_1) \) where \( U_2 = U_2(\dot{U}_2(w)) = s^{-n}e^* \dot{V}_1 \)\( \dot{U}_2 = -(n-1)(Ds/s)(1 + h_2) \) with \( h_2 \ll 1 \), and proceeding exactly as above, we obtain \( \dot{U}_2 \dot{U}_1(y/f) = Ws^{-n}e^* \dot{V}_2 \dot{V}_1(y) \) where \( w' \sim 1/n \) and \( \dot{V}_2 = W(1 - h_2(1 - 1)/fW) \). Continuing in this way, we eventually transform \( \dot{U}_n \cdots \dot{U}_1(y/f) \) into \( w''e^* \dot{V}_n \cdots \dot{V}_1(y) \), where \( w'' \sim 1/n! \) and \( \dot{V}_i - W < W/fW \) for each \( i \). This gives us \( L = uV_n \cdots \dot{V}_1 \), concluding the proof.

9. Lemma. If \( V \sim W \) and \( V - W < W/fW \) (equivalently: \( V - W < V/fV \)), then \( L \) is regular with respect to \( W \) iff \( L \) is regular with respect to \( V \).

Proof. Assume \( V - W < W/fW \). Then \( W = (V/W)V + (1 - V/W) = (1 + g)V - g \)

where \( g = (V - W)/W \). Thus \( \dot{W} \) is regular with respect to \( V \). One proves by induction that the operators \( (\dot{W})^p \) \( (p = 2, 3, \cdots) \) are regular with respect to \( V \): If true for \( p = k - 1 \), we have \( (\dot{W})^{k-1} = (\dot{V})^{k-1} + \sum_{i=0}^{k-1} F_i(\dot{V})^i \) with \( F_i < (fV)^{i-1} \) and

\[
(\dot{W})^k = ((1+g) \dot{V} - g) (\dot{V})^{k-1} + \sum_{i=0}^{k-1} F_i(\dot{V})^i).
\]

That the latter is regular with respect to \( V \) follows from the fact that \( \dot{V}(F_i(\dot{V})^i)(y) = F_i(\dot{V})^{i+1}(y) - (DF_i/V)(\dot{V})^i(y) \) where \( F_i < (fV)^{i+1} \) by the induction hypothesis, and \( DF_i/V < (fV)^{i+k} \). The latter inequality is seen by writing \( F_i = h(fV)^{i+1} \) with \( h \ll 1 \) and noting that \( Dh/V < (fV)^{-1} \) (because \( Dh < x^{-1}(\log x) \cdots (\log x)^{-1} \) if \( h \ll 1 \)). The regularity of \( L \) with respect to \( V \) follows readily from the regularity of the \( (\dot{W})^p \) with respect to \( V \) and the assumption that \( L \) is regular with respect to \( W \). The converse follows from the symmetry of \( V \) and \( W \) in this discussion.
10. **Lemma.** If \( L = u \prod_{i} V_{i} \) where \( u \sim 1 \), \( V_{1} \sim V_{2} \sim \cdots \sim V_{n} \sim W \) and \( V_{i} - W < W/fW \) for each \( i \), then \( L \) is regular with respect to \( W \).

**Proof.** With the operators \( \hat{V}_{i} \) expressed in the form \((1 + g_i) \hat{W} - g_i\) where \( g_i = 1 - W/V_i < 1/fW \), the proof of this lemma closely parallels the proof of Lemma 9.

11. **Lemma.** If \( L \) is regular with respect to \( W \), and \( V \sim W \) but \( V \sim W < W/fW \), then \( L \) is not regular with respect to \( V \).

**Proof.** Suppose \( L \) is regular with respect to \( V \). As a consequence of Lemmas 5 and 6, there exist solutions \( y_0 < y_1 < \cdots < y_{n-1} \) with \( y_i \sim \exp fV_i \) and \( V_i - W \sim W/fW \) for \( i > 0 \), while \( W_0 - W < W/fW \). At the same time we have solutions \( \tilde{y}_0 < \tilde{y}_1 < \cdots < \tilde{y}_{n-1} \) with \( \tilde{y}_i \sim \exp fV_i \), \( V_i - W \sim W/fW \) for \( i > 0 \), while \( V_0 - V < V/fV \). Since the \( y_i \) and the \( \tilde{y}_i \) constitute two bases for the space of solutions of \( L(y) = 0 \), it follows readily that \( y_0 \approx \tilde{y}_0 \), whence \( y_0/\tilde{y}_0 \approx 1 \) and \( \log(y_0/\tilde{y}_0) \approx 1 \). But \( y_0/\tilde{y}_0 \approx \exp f(W_0 - V_0) \), and \( W_0 - V_0 = (W_0 - W) + (W - V) + (V - V_0) \sim W/fW \). From this it follows that \( \log(y_0/\tilde{y}_0) \approx f(W_0 - V_0) > 1 \), a contradiction. Therefore \( L \) is not regular with respect to \( V \).

12. **Remark.** There exist unimajoral operators \( L \) of the form \((W)^n + \sum_{i=0}^{n} E_i(W)^i\) such that \( L \) is not regular with respect to \( V \) for any \( V \).

For example, \((1 - (1 + 1/x)^{-1}D)(1 - D)\) can be expressed in the form \((W)^2 + E_1W + E_0\), with \( E_0, E_1 < 1 \), for \( W = 1 \) (indeed, for any \( W \sim 1 \)). That this \( L \) is not regular with respect to \( V \) for any \( V \sim 1 \) may be seen as follows: Suppose \( L \) is regular with respect to \( V \sim 1 \). Then by Lemma 5 there exist \( W_0 \) and \( W_1 \) with \( W_0 - V \sim V/fV \) and \( W_1 - V \sim V/fV \) such that the solutions of \( L(y) = 0 \) are generated by \( y_0 \sim \exp fW_0 \) and \( y_1 \sim \exp fW_1 \). In this case \( W_1 - W_0 \sim V/fV \) and \( V/fV \sim 1/x \), so \( y_1/y_0 \sim e^f \) where \( f \sim \log x \), and \( \log(y_1/y_0) \sim \log x \). But one sees directly that the solutions of \( L(y) = 0 \) are generated by \( e^f \) and \( x^2e^x \), so \( y_1/y_0 \) must be \( = x^2 \), whence \( \log(y_1/y_0) \sim \log x \), a contradiction.

**References**

