METRIC-DEPENDENT DIMENSION FUNCTIONS

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The starting point of this study was an attempt to solve a problem proposed by G. M. Rosenstein [3] in his doctoral dissertation.

ROSENSTEIN’S PROBLEM. Suppose that $R$ is a metrizable space, $\dim R \geq n$ (covering dimension), and $\rho$ is any compatible metric for $R$. Then is it true that there exist $n$ pairs of closed sets $C_1, C'_1, \ldots, C_n, C'_n$ such that (i) $\rho(C_i, C'_i) > 0$ for all $i$ and (ii) if for each $i$, $B_i$ is a closed set separating $C_i$ and $C'_i$, then $\bigcap_{i=1}^{n} B_i \neq \emptyset$?

Remark. If, in this problem, “(i) $\rho(C_i, C'_i) > 0$” is replaced by “(i*) $C_i \cap C'_i = \emptyset$,” one obtains a characteristic property for metric spaces $R$ such that $\dim R \geq n$. (See [1, Remark, for the separable case, p. 78].)

Our main result is an example which shows that the answer to the problem is in the negative. In the notation given below, our example is a metric space $(R, \rho)$ such that $\dim R = 2$ and $d_2(R, \rho) = 1$. For $n = 1$ the problem is answered in the affirmative.

Definition of $d_n(R, \rho)$. Let $(R, \rho)$ be a metric space with metric $\rho$. We define $d_n(R, \rho)$ inductively as follows: For the empty set $\emptyset$, $d_1(\emptyset, \rho) = -1$. If for every pair of closed subsets $F, H$ of $R$ with $\rho(F, H) > 0$ there exists an open set $G$ with $F \subset G \subset R - H$ and with $d_1(G - G, \rho^*) \leq n - 1$, where $\rho^*$ is the restriction of $\rho$ to $G - G$, then we say $d_1(R, \rho) \leq n$. If there is no such integer $n$, then we say $d_1(R, \rho) = \infty$.

Theorem. For any metric space $(R, \rho)$ we have $d_1(R, \rho) = \text{Ind } R$, where $\text{Ind } R$ is the large inductive dimension of $R$ defined by means of sets separating disjoint closed pairs of subsets.

Proof. It is evident that $d_1(R, \rho) \leq \text{Ind } R$. When $d_1(R, \rho) = \infty$, it is also evident that $d_1(R, \rho) \geq \text{Ind } R$. Hence we suppose that $d_1(R, \rho) \leq n$ and make the induction assumption that $d_1(R', \rho') \geq \text{Ind } R'$ for any metric space $(R', \rho')$ with $d_1(R', \rho') \leq n - 1$. Let $H$ and $F$ be disjoint closed subsets of $R$. Put $D_i = \{x : \rho(x, H) < 1/i\}$, and $E_i = \{x : \rho(x, F) < 1/i\}$, $i = 1, 2, \ldots$. Then there exist open sets $M_i$ with $H \subset M_i \subset D_i$, $d_1(M_i - M_i, \rho) \leq n - 1$, $i = 1, 2, \ldots$, and open sets $N_i$ with $F \subset N_i \subset E_i$, $d_1(N_i - N_i, \rho) \leq n - 1$, $i = 1, 2, \ldots$. Put $G_i = M_i - N_i$, $i = 1, 2, \ldots$, and $G = \bigcup_{i=1}^{\infty} G_i$. Then it can easily be seen that $H \subset G \subset R - F$.

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Since $G_i - G_i \subseteq (M_i - M_i) \cup (N_i - N_i)$, we have $\text{Ind}(G_i - G_i) \leq n - 1$ by the induction assumption. Since $\{G_i : i = 1, 2, \ldots \}$ is locally finite at every point of $R - (H \cup F)$ and $G-G$ is contained in $R - (H \cup F)$, we have $G-G \subseteq \bigcup \{G_i - G_i : i = 1, 2, \ldots \}$. Every point $x$ of $G-G$ has a relative neighborhood $U(x)$ which is contained in the sum of a finite number of elements of $\{G_i - G_i : i = 1, 2, \ldots \}$. Hence $\text{Ind} \ U(x) \leq n - 1$ and we have $\text{Ind}(G-G) \leq n - 1$ by the local dimension theorem. Thus we know $\text{Ind} \ R \leq n$ and the theorem is proved.

**Corollary.** If $R$ is a metrizable space with $\dim R = 1$ and $\rho$ is any compatible metric for $R$, then there exists a pair of closed sets $C$ and $C'$ with $\rho(C, C') > 0$ such that the empty set cannot separate $C$ and $C'$.

**Definition of $d_2(R, \rho)$.** Let $(R, \rho)$ be a metric space with metric $\rho$. We write $d_2(\emptyset, \rho) = -1$. If there exists a greatest integer $n$ such that there exist $n$ pairs $C_i, C'_i, \ldots, C_n, C'_n$ such that (i) $\rho(C_i, C'_i) > 0$ and (ii) if for each $i$, $B_i$ is a closed set separating $R$ between $C_i$ and $C'_i$, then $\bigcap_{i=1}^n B_i = \emptyset$, then we say $d_2(R, \rho) = n$. Otherwise $d_2(R, \rho) = \infty$.

Rosenstein's problem may now be stated as follows: Is it true that $d_2(R, \rho) = \dim R$?

**Example.** In the closed 3-cell $I^3$ we define a countable disjoint collection $\mathcal{A} = \{A_i : i = 1, 2, \ldots \}$, set $A = \bigcup \mathcal{A}$ and $R = I^3 - A$. We prove that $\dim R = 2$ and $d_2(R, \rho) = 1$, where $\rho$ is the Euclidean metric on $I^3$. Let $\mathcal{U} = \{U_1, U_2, \ldots \}$ be any countable base for the topology of $I^3$. The set of all finite unions of elements of $\mathcal{U}$ is countable; hence so is the set of all quadruples of such unions and, a fortiori, any subset. Thus there exists a countable sequence $Q_1, Q_2, \ldots$, such that for all $i$

1. $Q_i = \{B_i, C_i, D_i, E_i\}$,
2. each of $B_i, C_i, D_i, E_i$ is a finite union of elements of $\mathcal{U}$,
3. $\rho(B_i, C_i) > 0$ and $\rho(D_i, E_i) > 0$, and
4. if $Q' = \{B', C', D', E'\}$ is any quadruple satisfying (2) and (3) then for some $i$ we have $Q' = Q_i$.

For each $i$ we define closed sets $M_i$ and $N_i$ such that $M_i$ separates $B_i$ and $C_i$, $N_i$ separates $D_i$ and $E_i$, and we set $A_i = M_i \cap N_i$. We want $\mathcal{A} = \{A_i : i = 1, 2, \ldots \}$ to be a disjoint collection. Let $\varepsilon = \min \{\rho(B_i, C_i), \rho(D_i, E_i)\}$. Let $\pi, \pi_\varepsilon, \pi_\varepsilon, \pi_\varepsilon, \ldots$ be a monotonically increasing sequence of prime numbers such that for all $i$, $1/\pi_i < \varepsilon/\sqrt{3}$. Divide $I^3$ into $\pi_i^3$ small closed cubes whose edges have length $1/\pi_i$, let $H_i$ denote the union of all such cubes that intersect $B_i$, and let $M_i = \text{boundary of } H_i$. Similarly, using $\pi_\varepsilon$ instead of $\pi_i$. 

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let $K_i$ be the union of all small cubes intersecting $D_i$ and let $N_i$ = boundary of $K_i$. Set $A_i=M_i\cap N_i$. If $x\in A_i$, then one coordinate of $x$ is of the form $a/\pi_i$ and one coordinate is of the form $b/\pi'_i$, where $a$ and $b$ are integers, $0<a<\pi_i$, $0<b<\pi'_i$.

**Assertion 1.** For all $i$, $M_i$ is a closed set separating $B_i$ and $C_i$ and $N_i$ is a closed set separating $D_i$ and $E_i$.

**Assertion 2.** For all $i$, $A_i$ is closed and $\dim A_i \leq 1$.

**Proof.** If $a$ and $b$ are positive integers with $0<a<\pi_i$, $0<b<\pi'_i$, then $a/\pi_i \neq b/\pi'_i$; hence if $x\in A_i$, then at least two of its coordinates are rational. By [1, Example III 6, p. 29] this shows that $\dim A_i \leq 1$.

**Assertion 3.** If $i \neq j$, then $A_i \cap A_j = \emptyset$.

**Proof.** Assume there exists $x \in A_i \cap A_j$. Then there exist four integers $0 < a < \pi_i$, $0 < b < \pi'_i$, $0 < c < \pi_j$, $0 < d < \pi'_j$ such that $x$ must have coordinates equal to each of $a/\pi_i$, $b/\pi'_i$, $c/\pi_j$, $d/\pi'_j$, four distinct numbers. But $x$ has only 3 coordinates.

**Assertion 4.** $d_2(R, \rho) \leq 1$.

**Proof.** Let $B, C, D, E$ be any relatively closed sets in $R$ such that $\rho(B, C) > 0$, $\rho(D, E) > 0$. Then their closures in $I^n$ are compact and for some $i$ we have $B \subseteq B_i$, $C \subseteq C_i$, $D \subseteq D_i$ and $E \subseteq E_i$; so $M_i$ separates $B$ and $C$ in $I^n$, $N_i$ separates $D$ and $E$ in $I^n$. But then $M_i \cap R$ and $N_i \cap R$ are corresponding separating sets in $R$, and their intersection is vacuous since $M_i \cap N_i = A_i \subset I^n - R$.

**Assertion 5.** $\dim R \geq 2$.

**Proof.** This follows from Brouwer's theorem on invariance of domain, since it can easily be seen that $A$ is dense in $I^n$.

**Assertion 6.** $\dim R \geq 2$.

To prove this assertion we need the following two lemmas.

**Lemma 1.** Let $S$ be a subset of the closed $n$-cell $I^n$ with $\dim S \leq n - 2$. Then for any points $p$ and $q$ in $I^n - S$ there exists a continuum $K$ such that (i) $K$ is contained in $I^n - S$ and (ii) $K$ contains $p$ and $q$. (See [4].)

**Lemma 2.** A continuum cannot be decomposed into a countably infinite or finite (but more than one) union of pairwise disjoint closed subsets. (See [2, Theorem 44, p. 30].)

**Proof of assertion 6.** Assume that $\dim R < 2$. If $A$ is closed, then $R$ is an $F_\sigma$ and $\dim I^n \leq \max(\dim A, \dim R) < 2$, which is impossible. Thus there exist integers $i$ and $j$ with $i \neq j$ such that $A_i \neq \emptyset$, $A_j \neq \emptyset$. Take $p \in A_i$, $q \in A_j$. By Lemma 1 there exists a continuum $K$ such that (i) $\{p, q\} \subset K$ and (ii) $K \subset I^n - R = A$. Thus $K = \bigcup_{i \neq j}^n (K \cap A_i)$, a countable union of pairwise disjoint closed sets at least two of which
are not vacuous. But this is impossible by Lemma 2, so the proof of
Assertion 6 is completed.

**Assertion 7.** \(d_2(R, \rho) \geq 1\).

**Proof.** Since \(\dim R = 2\), \(d_2(R, \rho) \neq -1\). It is obvious that \(d_2(R, \rho) = 0\) if and only if \(d_1(R, \rho) = 0\). On the other hand we already know that \(d_1(R, \rho) = \dim R\) by the above theorem. Thus \(d_2(R, \rho) < 1\) contradicts \(\dim R = 2\).

**Remark.** By a construction quite similar to that just given, one may start with \(I^n\) for \(n > 3\), and define a subset \(R\) such that \(\dim R = n - 1\) and \(d_2(R, \rho) \leq n/2\).

**Problem.** Is it true that, given a metric space \((R, \rho)\), \(\dim R \geq n\) implies that \(d_2(R, \rho) \leq n/2\)?

**References**


