POWERS IN EIGHTH-GROUPS

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1. Introduction. The purpose of this paper is to give an algorithm which decides whether or not an element in an eighth-group is a power. A group $G$ is an eighth-group if it is finitely presented in the form

$$G = gp(a_1, \ldots, a_n; R_1(a_\lambda) = 1, \ldots, R_m(a_\lambda) = 1),$$

where (i) each defining relator is cyclically reduced and (ii) if $B_i$ and $B_j$ are cyclic transforms of $R_i$ and $R_j$, then less than one-eighth of the length of the shorter one cancels in the product $B_i^{\pm 1}B_j^{\pm 1}$, unless the product is unity. The notation in this paper is the same as that in [3]. Note that Lemma 3 and Lemma 4 in [3] hold for eighth-groups.

Reinhart [4] gives an algorithm to decide, among other things, whether or not elements in certain Fuchsian groups are powers. Note that the Fuchsian group $F(p; n_1, \ldots, n_d; m)$, see Greenberg [1], is an eighth-group if

$$4p + d + m, n_1, \ldots, n_d > 8.$$ 

Hence our algorithm holds for a somewhat wider class of groups and, furthermore, is purely algebraic.

Remark. Given any word $V$ in a finitely presented group, it is possible to find a cyclically fully reduced word $V^*$ conjugate to $V$ by writing the word $V$ in a circle and then reducing. Such a word $V^*$ will be called a \textit{reduced cyclic transform} of $V$.

2. The algorithm. First we prove a lemma about eighth-groups $G$. Here $r$ denotes the length of the largest defining relator in $G$. 

Received by the editors April 3, 1964.
Lemma. Let \( W \) be cyclically fully reduced, let \( W \) be conjugate to \( V \), and let \( l(V) = n \). Then \( l(W) \leq r^2 + rn \).

Proof of Lemma. By Greendlinger’s Basic Theorem in [2, p. 643], there exist reduced cyclic transforms \( W^* \) and \( V^* \) of \( W \) and \( V \) such that \( W^* = T^{-1}V^*T \), where \( l(T) < r/8 \) and \( l(V^*) \leq l(V) \). Hence

\[
l(T^{-1}V^*T) < r/8 + n + r/8 < r + n.
\]

Consequently, by Lemma 3 in [3],

\[
l(W^*) \leq rl(T^{-1}V^*T) \leq r^2 + rn.
\]

But \( W \) cyclically fully reduced implies \( l(W) = l(W^*) \). Hence the lemma is true.

Suppose, now, that an arbitrary word \( W \neq 1 \) in an eighth-group is a power, say \( W = V^m \) and \( l(W) = n \). Let \( A \) be a reduced cyclic transform of \( V \); then \( W \) is conjugate to \( A^m \). Lemma 4 in [3] implies that \( A^m = B \), where \( B \) is cyclically fully reduced and where (i) \( l(B) \geq m \), and (ii) \( l(B) \geq l(A) - r \). Accordingly, our lemma above implies

\[
(1) \quad m \leq l(B) < r^2 + nr,
\]

\[
(2) \quad l(A) \leq l(B) + r < r^2 + nr + r.
\]

The above discussion proves the following

Theorem. Let \( W \neq 1 \) be an arbitrary word in an eighth-group \( G \) where \( l(W) = n \) and \( r \) is the length of the largest defining relator in \( G \). Then \( W \) is a power if and only if \( W \) is conjugate to \( A^m \) where \( m \) and \( A \) satisfy (1) and (2).

Since the conjugacy problem has been solved for eighth-groups by Greendlinger in [2], and since there exist only a finite number of words in any given length, the above theorem gives us our algorithm.

Bibliography


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