ON A PROBLEM CONNECTED WITH THE VANDERMONDE DETERMINANT

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Let m be a positive integer. We denote by \( a = (a_0, a_1, \cdots, a_m), \)
\( x = (x_0, \cdots, x_{m-1}) \) vectors with real components. Define

\[
U = \{ x \mid \text{all } |x_i| \leq 1, x_i \neq x_j, i \neq j \},
\]

\[
A^* = \{ a \mid P^*(x, a) \neq 0 \text{ for every } x \in U \}.
\]

This paper is devoted to the problem of characterizing \( A^* \), a problem which is of interest in the theory of the design of statistical experiments (see [1]) and is perhaps of interest per se.

When \( m = 1 \), it is trivial that \( A^* = \{ (a_0, a_1) \mid |a_0| < |a_1| \} \). We hereafter assume \( m > 1 \). Define

\[
A(c) = A^* \cap \{ a \mid a_m = c \}.
\]

We shall prove the following:

(I) \( A(1) \) is a set \( A \) described precisely below.

(II) For \( c \neq 0 \), we have \( (a_0, a_1, \cdots, a_{m-1}, c) \in A(c) \) if and only if \( (a_0/c, \cdots, a_{m-1}/c, 1) \in A \).

(III) \( A(0) \) is empty.

Of these, (II) is obvious, and (III), which we shall prove in the paragraph after next, is only slightly less so. The problem is to describe \( A \).

Let \( S_0(x) = 1 \) and

\[
P^*(x, a) = \begin{vmatrix}
1 & 1 & \cdots & 1 & a_0 \\
x_0 & x_1 & \cdots & x_{m-1} & a_1 \\
\frac{x_0}{2} & \frac{x_1}{2} & \cdots & \frac{x_{m-1}}{2} & a_2 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
x_0 & x_1 & \cdots & x_{m-1} & a_m \\
\end{vmatrix},
\]

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(2) \[ S_j(x) = (-1)^j \sum x_{i_1} x_{i_2} \cdots x_{i_j}, \]
where the summation is with respect to \( i_1, \ldots, i_j, \) \( 0 \leq i_1 < i_2 < \cdots < i_j < m. \) Let

(3) \[ P(x, a) = \sum_{j=0}^{m} a_{m-j} S_j(x). \]

Expanding the determinant \( P^* \) of (1) in minors of the last column shows that

\[ P^*(x, a) = P(x, a) K(x), \]

where the polynomial \( K \) is the Vandermonde determinant which is the minor of \( a_m \) in the determinant \( P^* \) of (1); thus, \( K \) is never zero on \( U. \) Hence in the definition of \( A^* \) we may replace \( P^*(x, a) \) by \( P(x, a). \)

We shall now prove that \( A(0) \) is empty. Let \( a \) be any \((m+1)\)-vector with \( a_m = 0. \) If all \( a_j = 0, \) clearly \( a \in A^*. \) Suppose then that \( j_0 \)
is the smallest integer for which \( a_{m-j_0} \neq 0, \) \( 0 < j_0 \leq m. \) We shall find two points \( x' \) and \( x'' \) in the convex subset \( T \) of \( U \) where \( -1 \leq x_0 < x_1 < \cdots < x_{m-1} \leq 1 \) such that

(4) \[ (-1)^{j_0} a_{j_0} P(x', a) < 0 < (-1)^{j_0} a_{j_0} P(x'', a), \]

from which it follows that \( P(x, a) = 0 \) for some \( x \) in \( T; \) it follows that \( A(0) \) is empty. Let \( x'' = (e, 2e, \cdots, me) \) with \( 0 < e < 1. \) Each term in the sum in (2) is then positive, and thus \( (-1)^j a_j S_j(x'') > e^{1j} \) while \( S_j(x'') = o(e^j) \) as \( e \to 0 \) if \( j > j_0. \) Hence, for \( e \) sufficiently small, the second half of (4) follows from (3). The first half of (4) follows similarly if \( j_0 \) is odd upon taking \( x' = (-me, -(m-1)e, \cdots, -e), \) and if \( j_0 \) is even upon taking \( x' = (-m-1)e, -(m-2)e, \cdots, -e, 1). \)

Let

(5) \[ Q_h(x) = \sum (1 - x_{i_1})(1 - x_{i_2}) \cdots (1 - x_{i_h})(1 + x_{i_{h+1}}) \cdots \\
\quad \cdot (1 + x_{i_m}) \binom{m}{h}, \quad 0 \leq h \leq m, \]

where the summation is over the \( \binom{m}{h} \) choices of the disjoint sets \((i_1, i_2, \cdots, i_h) \) and \((i_{h+1}, \cdots, i_m) \) of distinct integers between 0 and \( m-1, \) inclusive. Define the points

(6) \[ a^{(h)} = (a_0^{(h)}, \cdots, a_m^{(h)}), \quad 0 \leq h \leq m, \]

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We note that \(a^{(h)}_m = 1\) for all \(h\), and that
\[
\begin{align*}
a^{(0)} &= ((-1)^m, (-1)^{m-1}, \ldots, (-1)^0), \\
a^{(m)} &= (1, 1, \ldots, 1).
\end{align*}
\]

We will now prove:

**Theorem.** For \(m > 1\), \(A\) is the closed \(m\)-simplex with extreme points \(a^{(0)}, \ldots, a^{(m)}\), minus the (closed) edge which connects \(a^{(0)}\) and \(a^{(m)}\).

Let \(\overline{U}\) be the closure of \(U\). If \(x\) is a vertex of \(\overline{U}\) with \(r\) coordinates \(-1\) and \((m-r)\) coordinates \(+1\), then
\[
Q_h(x) = \begin{cases} 2^{-r} & \text{if } h = r, \\ 0 & \text{if } h \neq r. \end{cases}
\]

Hence the \(Q_h\) are linearly independent functions. Moreover, each \(Q_h(x) \geq 0\) on \(U\). If \(h \neq 0\) or \(m\), \(Q_h(x) > 0\) on \(U\), since at most one \(x_i\) can take the value \(+1\) and at most one \(x_i\) can take the value \(-1\) on \(U\).

According to (3), \(P(x, a)\) is linear in \(a\). According to (7), \(P(x, a^{(h)}) = Q_h(x)\). Combining these facts, if \(\alpha_0, \ldots, \alpha_m\) are real numbers,
\[
P \left( x, \sum_{h=0}^{m} \alpha_h a^{(h)} \right) = \sum_{h=0}^{m} \alpha_h Q_h(x).
\]

Since the \(Q_h\) are linearly independent, it follows from (7) that the vectors \(a^{(i)}, 0 \leq i \leq m\), are linearly independent. Hence, \(a^{(0)}, \ldots, a^{(m)}\) lie in the hyperplane \(a_m = 1\), but in no other hyperplane. Hence, any point \(a\) in the hyperplane \(a_m = 1\) can be written in a unique way as
\[
a = \sum_{h=0}^{m} \alpha_h a^{(h)},
\]
where \(\sum \alpha_h = 1\). Suppose now that \(\alpha_h \geq 0\) for all \(h\), and \(\alpha_h > 0\) for some \(h \neq 0\) or \(m\). Then, from (8) and the paragraph below the statement of the theorem, we have \(P(x, a) > 0\) for all \(x\) in \(U\). From this it follows that \(A\) is at least as large as stated in the theorem. It remains to prove that it is no larger.

If \(\alpha_0 + \alpha_m = 1\) and all other \(\alpha_h = 0\), we have \(P(x, a) = 0\) for any \(x\) in \(U\) such that \(x_0 = 1, x_1 = -1\). Hence the closed segment \([a^{(0)}, a^{(m)}]\) is not in \(A\).
Suppose now that $\alpha_{i_0} < 0$. If 0 is the origin then of course $P(0, a) = 1$. It follows from the paragraph below the statement of the theorem that there is an $x^*$ in $U$ such that (i) the half-open segment $(0, x^*]$ is in $U$, (ii) $Q_{i_0}(x^*) > \frac{1}{2}$, and (iii) for $j \neq i_0$, $Q_j(x^*) < |\alpha_{i_0}| / 2 \sum |\alpha_i|$. From (8) we have $P(x^*, a) < 0$. Hence, for some point $y \in (0, x^*)$ (so that $y \in U$) we must have $P(y, a) = 0$. Hence $a$ cannot be in $A$. The proof of the theorem is complete.

We remark that the points $a$ of the form $(b^m, b^{m-1}, \ldots, b^1, 1)$ with $|b| < 1$ are easily verified to be in $A$; this is of interest in applications.

Reference


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