Let $R$ be an associative ring in which an identity element is not assumed. A right quotient ring of $R$ is an overring $S$ such that for each $a \in S$ there corresponds $r \in R$ such that $ar \in R$ and $ar \neq 0$. A theorem of R. E. Johnson [1] states that $R$ possesses a right quotient ring $S$ which is a (von Neumann) regular ring if and only if $R$ has vanishing right singular ideal. In this case $R$ possesses a unique (up to isomorphism over $R$) maximal right quotient ring $S$, and $S$ is regular and right self-injective (Johnson-Wong [1]). It is easy to see that $S$ is the injective hull of $R$, considering both rings as right $R$-modules in the natural way. Thus, each right ideal $I$ of $R$ has an injective hull $I_R$ contained in $S$. In this notation, $S = \hat{R}_R$, and we use $\hat{R}$ to denote the maximal right quotient ring of $R$ hereafter. By the results of Johnson [2], $I_R$ can be characterized in two ways:

(a) $I_R$ is the unique maximal essential extension of $I$ contained in the right $R$-module $\hat{R}$.

(b) $I_R$ is the principal right ideal of $\hat{R}$ generated by $I$.

Since $I_R$ is therefore a right ideal of $\hat{R}$, $\Delta = \text{Hom}_R (\hat{I}_R, \hat{I}_R)$ is defined. Setting $\Gamma = \text{Hom}_R (I, I)$, one of our main results (Theorem 2) states that $\hat{I} = \text{Hom}_R (\hat{I}_R, \hat{I}_R) = \Delta$. This means that $\Gamma$ has vanishing right singular ideal, and that $\Delta$ is the maximal right quotient ring of $\Gamma$.

Since $I_R$ is a principal right ideal in the regular ring $\hat{R}$, there exists an idempotent $e \in \hat{R}$ such that $I_R = e\hat{R}$. Then, of course, $\Delta \cong e\hat{R}e$, and it is natural to investigate the relationship between $e\hat{R}e$ and $K = e\hat{R}e \cap R$. In general, it is too much to hope that $e\hat{R}e = \hat{R}$, since it is possible that $K = 0$ for some nonzero $e \in \hat{R}$. Nevertheless, under the assumption that $\hat{R}$ is also a left quotient ring of $R$, or in case $e$ is a primitive idempotent satisfying $e\hat{R}e \cap R \neq 0$, we establish (Theorem 3) that $K$ has vanishing right singular ideal, and that $e\hat{R}e = \hat{K}$ (= the maximal right quotient ring of $K$.) In any case, for any nonzero idempotent $e \in \hat{R}$, $e\hat{R}e$ is the maximal right quotient ring of $e\hat{R}$.

Since we are not restricting ourselves to rings with identity, we say

---

1 Part of this paper was presented at the Stockholm International Congress, 1962, under the title Classical and maximal quotient rings.

The authors wish to acknowledge partial support by the National Science Foundation.
that an arbitrary (right) module $M_R$ over a ring $R$ is injective in case it
possesses the following property: if $A_R$ is any module, and if $B_R$
is any submodule, then any homomorphism $B_R \rightarrow M_R$ can be extended
to a homomorphism $A_R \rightarrow M_R$. In case $R$ has an identity element,
then a unital module $M_R$ is $\mu$-injective in case it has the property
above with $A_R$ ranging over all unital modules. It is easy to see that
a unital module $M_R$ is injective if and only if it is $\mu$-injective (cf.\nFaith-Utumi [1]). Baer's criterion (loc. cit.) states that a unital
module $M_R$ is injective if and only if it has the following property:
If $f$ is any module homomorphism of a right ideal $I$ of $R$ into $M$, then
there exists $m \in M$ such that $f(x) = mx \forall x \in I$. Call this latter property
of a module $M_R$ Baer's condition. It is known (loc. cit.) that if $M_R$
is an arbitrary injective module, then it satisfies Baer's condition.
Accordingly, if $S$ is any ring which is right self-injective, the identity
map $x \rightarrow x$ can be performed by a left multiplication by an element
$e \in S$ which is patently a left identity element of $S$. If $S$ is left-faith-
ful, any left identity is two-sided. In particular, any semiprime right
self-injective ring possesses an identity element. We use this fact
below.

**Theorem 1.** If $S$ is semiprime and right self-injective, then for any
idempotent $e \in S$, $eSe$ is semiprime and right self-injective.

**Proof.** Let $I$ be any right ideal of $eSe$ which is nilpotent of index
2. Then $(IS)^2 = (IS)(IS) = (I(eS))((eIS) = [I(eSe)]IS \subseteq I^2S = 0$. Thus,
$IS$ is a nilpotent right ideal of $S$, whence $IS = 0$ and $I = 0$. Since
$eSe$ does not contain nilpotent right ideals of index 2, it follows that
$eSe$ is semiprime.

Now let $I$ be any right ideal of $eSe$, let $x = \sum_{i=1}^{n} x_i s_i$, $x_i \in I$, $s_i \in S$,
i = 1, \ldots , n be any element of IS. Let $f \in \text{Hom}_{eSe}(I, eSe)$, and let $T$
denote the set of all elements $\sum_{i=1}^{n} f(x_i) r_i \in \sum_{i=1}^{n} f(x_i) S$, $r_i \in S$, $i = 1, \ldots , n$, such that $\sum_{i=1}^{n} x_i r_i = 0$. Clearly $T$ is a right ideal of $S$, and
$T \subseteq \sum_{i=1}^{n} f(x_i) S \subseteq eS$. Now if $t = \sum_{i=1}^{n} f(x_i) r_i \in T$, then

$$le = \left[ \sum_{i=1}^{n} f(x_i) r_i \right] e = \sum_{i=1}^{n} [f(x_i) r_i] e = \sum_{i=1}^{n} f(x_i) (er_i e)$$

$$= \sum_{i=1}^{n} f(x_i e r_i e) = f\left( \sum_{i=1}^{n} x_i r_i e \right) = f(0) = 0.$$

Thus $T^2 = (eT)^2 = 0$, and $T = 0$, since $S$ is semiprime. It follows that
$x = \sum_{i=1}^{n} x_i s_i = 0$ implies that $\sum_{i=1}^{n} f(x_i) s_i = 0$, so that the correspondence

$$f': x \rightarrow \sum_{i=1}^{n} f(x_i) s_i,$$
defined for any \( x \in I S \), is an element of \( \text{Hom}_S(IS, S) \). Since \( S_S \) is injective, and \( S \) is semiprime, \( S \) has an identity element, so \( S_S \) satisfies Baer's condition. Accordingly, there exists \( m \in S \) such that \( f'(x) = mx \forall x \in IS \). In particular, if \( x \in I \), then \( x = xe \), so that \( f'(x) = f(x)e = f(x) \). Thus, \( f(x) = (eme)x \forall x \in I \). Since \( eme \in eSe \), this shows that \( (eSe)_S \) satisfies Baer's condition, and \( eSe \) is therefore right self-injective.

The proof of the next theorem needs some facts about essential and rational extensions, and the proofs of these may be found in Findlay-Lambek [1] and Johnson-Wong [1]. Recall that \( M_R \) is an essential extension of a submodule \( N_R \) in case each nonzero submodule of \( M_R \) has nonzero intersection with \( N_R \); we symbolize this by \( (M \triangledown N)_R \); \( R(M \triangledown N) \) denotes the right-left symmetry for a left module \( R M \).

Then \( N_R \) is an essential submodule of \( M_R \). An essential left ideal \( I \) of \( R \) is a left ideal such that \( R \triangledown I \).

The singular submodule \( Z(M_R) \) is defined by:

\[
Z(M_R) = \{ x \in M \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R \}.
\]

We let \( Z_r(R) \) denote \( Z(R_R) \); it is an ideal, called the right singular ideal of \( R \).

A module \( M_R \) is rational over a submodule \( N_R \) in case it has the following property: if \( P \) is any module satisfying \( M \supseteq P \supseteq N \), and if \( f \in \text{Hom}_R(P, M) \), then \( f = 0 \) if and only if \( f(N) = 0 \). We let \( (M \triangledown N)_R \) denote a rational extension \( M \) of \( N \). Any rational extension is essential; moreover:

- If \( Z(N_R) = 0 \), then \( (M \triangledown N)_R \) if and only if \( (M \triangledown N)_R \).

We shall use the following characterization of rational extensions:

\( (M \triangledown N)_R \) if and only if for each pair \( x, y \in M \) with \( y \neq 0 \), there exist \( r \in R \) and an integer \( n \) such that

\[
xr + xn \in N \quad \text{and} \quad yr + yn \neq 0.
\]

We also need the following facts about quotient rings (Johnson [2]): Let \( R \) be such that \( Z_r(R) = 0 \), and adopt the notation of the introduction. For each right ideal \( I \) of \( R \), let \( \bar{I} = I_R \cap R \). Then \( \bar{I} \) is the unique maximal essential extension of \( I_R \) contained in \( R \); \( I \) is a closed right ideal of \( R \) in case \( I = \bar{I} \). The totality \( C_r(R) \) of closed right ideals of \( R \) is a complete lattice, and \( C_r(\hat{R}) \) is isomorphic to \( C_r(R) \) under the contraction map \( A \rightarrow A \cap R \). From what we already have said, \( C_r(\hat{R}) \) consists of the principal right ideals of \( \hat{R} \).

**Theorem 2.** Let \( R \) be a semiprime ring such that \( Z_r(R) = 0 \), let \( I \) be any right ideal of \( R \), let \( \bar{I}_R \) denote the principal right ideal of \( \hat{R} \) generated by \( I \), let \( \Gamma = \text{Hom}_R(I, I) \), and let \( \Delta = \text{Hom}_R(\bar{I}_R, \bar{I}_R) \). Then \( Z_r(\Gamma) = Z_r(\Delta) = 0 \), and \( \Delta = \bar{\Gamma} \) (= the maximal right quotient ring of \( \Gamma \)).
Proof. Set $S = R$. As remarked above, $\hat{I}_R$ is the injective hull of $I_R$ contained in $S_R$, and is the principal right ideal of $S$ generated by $I$. Since $S$ is regular, $\hat{I}_R = eS$, where $e = e^2 \in S$.

We first show that $\Delta = \text{Hom}_S(\hat{I}_R, \hat{I}_R)$ coincides with $\Omega = \text{Hom}_R(I, I)$. Clearly $\Omega \supseteq \Delta$. Conversely if $f \in \Omega$, and if $r \in \hat{I}_R \cap R = I$, then $f(r) = f(e)r$. If $x \in \hat{I}_R$, then $x_R = \{ t \in R \mid xt \in I \}$ is an essential right ideal of $R$. Now if $t \in x_R$, then

$$f(x)t = f(xt) = f(e)xt,$$

that is, $(f(x) - f(e)x)t = 0$. Thus, $[f(x) - f(e)x]x_R = 0$. Since $\hat{I}_R$ has zero singular submodule, we conclude that $f(x) = f(e)x \forall x \in \hat{I}_R$, and then clearly $f \in \Delta$. Thus establishes $\Omega = \Delta$.

Since $\hat{I}_R$ is the injective hull of $I_R$, it follows that each $\gamma \in \Gamma$ has an extension $\hat{\gamma} \in \Delta = \Omega$, and $\hat{\gamma}$ is unique, since $\hat{I}_R$ is rational over $I_R$. Clearly $\{ \gamma \in \Gamma \mid \gamma \in \Gamma \}$ is a subring of $\Delta$ isomorphic to $\Gamma$ under $\gamma \leftrightarrow \hat{\gamma}$. Henceforth, consider $\Gamma$ as a subring of $\Delta$.

Now $\Delta$ is isomorphic to the ring $eS_\Gamma$. If $I_L$ denotes the totality of left multiplications $aL$ of $I$ by elements $a \in I$,

$$a_L: x \rightarrow ax \quad x \in I,$$

then $I_L$ is a subring of $\Gamma$, and the natural isomorphism $\Delta \cong eS_\Gamma$ maps $I_L$ onto $eI$ and maps $\Gamma$ onto a subring $\Gamma_e$ of $eS_\Gamma$. Since $\Gamma_e \supseteq eI$, in order to show that $(eS \cup eI)_R$, it suffices to show that $(eS \cup eI)_{eI}$.

Now let $0 \neq \delta \in eS$. Since $(eS \cup eI)_R$, there exists $r \in R$ such that $0 \neq \delta r \in I$. Since $\delta r = \delta(er)$, it follows that $er \neq 0$. By the same reasoning, since $(eS \cup eI)_R$, there exists $s \in R, n \in \mathbb{Z}$, such that $u = er(s + n) \in I$, and $w = \delta r(s + n) \neq 0$. Since $\delta r \in I$, it follows that $w \in I$. Since $R$ is semiprime, $R$ is left-faithful, hence $wR \neq 0$ and also $(wR)^2 \neq 0$. Therefore, one can choose $t \in R$ such that $w' = wt$ satisfies $w' = w$ $\neq 0$. Then $w' = \delta u'$, where $u' = ut \in I$ and $w' \in I$, and

$$0 \neq w'' = \delta u'',$$

with $u'' = u'e \in eI$ and $w'' \in eI$. Thus, $(eS \cup eI)_{eI}$ as asserted. Hence $eS_\Gamma$ is a right quotient ring of $\Gamma_e$ and $\Delta$ is a right quotient ring of $\Gamma$. Now $S$ is regular (hence semiprime), so that $\Delta = eS_\Gamma$ is right self-injective by Theorem 1. Thus, $\Delta$ is a maximal right quotient ring of $\Gamma$. Since $\Delta(\cong eS_\Gamma)$ is regular, $Z_r(\Gamma) = Z_r(\Delta) = 0$.

Theorem 3. Let $R$ be any semiprime ring satisfying $Z_r(R) = 0$, and let $e$ be any idempotent in $S = \bar{R}$. Then:

(1) $eS_\Gamma = \bar{R}$, where $K = eRe$.

(2) If $S$ is also a left quotient ring of $R$, then $eS_\Gamma = \bar{R}$, where $K = eS_\Gamma \cap R$, and $eS_\Gamma$ is also a left quotient ring of $K$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(3) If \( e \) is a primitive idempotent, and if \( eSe \cap R \neq 0 \), then \( eSe \) is the right quotient field of \( K = eSe \cap R \).

**Proof.** Let \( B = (eS \cap R) + (1-e)S \cap R \). The lattice isomorphism \( C_r(R) \cong C_r(R) \) implies that
\[
[eS \cap R]_R \quad \text{(resp. \( (1-e)S \cap R]_R \)),}
\]
and it follows that \((S \cap B)_R\). For each \( x \in S \), \( x_B = \{ b \in B \mid xb \subseteq B \} \) is a right ideal of \( B \), and \((R \cap x_B)_R\).

Now choose \( \delta \in eSe \) and \( \delta \neq 0 \). Since \( \delta_B \) is an essential right ideal of \( R \), then \( \delta R \neq 0 \) (since \( Z(S_B) = Z(R_R) = 0 \)). By semiprimeness of \( R \), we see that \((\delta \delta_R) \neq 0 \), so that \( \delta \delta_R \neq 0 \). Hence we can choose \( b \in \delta_B \subseteq B \) such that \( \delta \delta_R = 0 \). (Note that \( eb \subseteq eS \cap R \).)

**Case (1).** Now \( \delta b \subseteq R \), and \( b \subseteq R \), and \( \delta be = \delta (eb) = e(\delta b)e \). Thus \( 0 \neq \delta (eb)e \subseteq eRe \). When \( eb \subseteq eRe \). This shows that \( eSe \) is a right quotient ring of \( eRe \).

**Case (2).** Since \( S \) is a regular ring which is both a right and left quotient ring of \( R \), necessarily \( Z_r(R) = 0 \). Then \( R \) is a rational extension of \( S \), and, moreover, \( (Se \cap R) \) is a rational extension of \( (S \cap R) \). The correspondence \( x \rightarrow x \delta be \), \( x \subseteq eSe \) is an element \( f \in \Hom_R(Se, Se) \), and \( f \neq 0 \) since \( e(\delta be) = \delta be \neq 0 \). It follows that \( f(Se \cap R) \neq 0 \), that is, that \( (Se \cap R) \delta be \neq 0 \) and \( (Se \cap R) \delta b \neq 0 \). By the semiprimeness of \( R \), \([Se \cap R] \delta b \neq 0 \), and so \( \delta b(Se \cap R) \neq 0 \). Hence choose \( u \in Se \cap R \) such that \( \delta bu \neq 0 \). Since \( b, \delta b, u \subseteq R \), then also \( \delta bu, bu \subseteq R \), and
\[
\delta bu = (eb)u(e) = e(\delta bu)e \subseteq eSe \cap R = K.
\]
Since \( eb \subseteq eS \cap R \), and \( u \subseteq eS \cap R \), then \( k = ebu \subseteq K \), so that \( 0 \neq \delta k = \delta bu \subseteq K \), with \( k \subseteq K \). Since \( \delta \) was an arbitrary nonzero element of \( eSe \), this proves that \( eSe \) is a right quotient ring of \( K \).

**Case (3).** \( eSe \) is a division ring and \( S \) is a right vector space over \( eSe \). Since \( Se \cap R \neq 0 \), and since \( 0 \neq \delta be \subseteq eSe \), it follows that \( (Se \cap R) \delta be \neq 0 \), and the rest of the proof proceeds as in the proof of (2).

In all cases we have deduced that \( eSe \) is a right quotient ring of \( K \) without resource to the fact that the right quotient ring \( S \) of \( R \) is maximal. Now in Case (2), \( Z_r(R) = 0 \) is a consequence of the fact that \( S \) is a regular ring which is a left quotient ring of \( R \). Hence, by symmetry, we conclude that \( eSe \) is a left quotient ring of \( K \) in this case.

Since \( eSe \) is regular along with \( S \), and since \( eSe \) is right self-injective by Theorem 1, we conclude that \( eSe = K \) in all cases, completing the proof.

**Remark.** It can be shown that \( K \) in (2) need not be semiprime.
We construct an example which shows that (2) and (3) of Theorem 3 fail under a weakening of the hypothesis.

Let $K$ be a right Ore domain which is not a left Ore domain, and let $k_1, k_2$ be nonzero elements of $K$ such that $K k_1 \cap K k_2 = 0$.

If $D$ denotes the right quotient division ring of $K$, then $S = D_2$, the full ring of all $2 \times 2$ matrices over $D$, is the classical, and maximal, right quotient ring of $R = K_2$. Let $\{e_{ij} \mid i, j = 1, 2\}$ denote matrix units in $S$, let $a = k_1^{-1} e_{11} + k_2^{-1} e_{12}$, and suppose $b \in S$ is such that $ba \in R$. Then $b = \sum_{i,j=1}^2 c_{ij} e_{ij}$, with $c_{ij} \in D$, $i, j = 1, 2$, and

$$ba = c_{11} k_1^{-1} e_{11} + c_{12} k_2^{-1} e_{12} + c_{21} k_1^{-1} e_{21} + c_{22} k_2^{-1} e_{22}.$$  

Since $ba \in K_2 = R$, necessarily

$$c_{11} k_1^{-1}, c_{12} k_2^{-1}, c_{21} k_1^{-1}, c_{22} k_2^{-1} \in K,$$

and then $c_{11}, c_{22} \in K k_1 \cap K k_2$. Since $K k_1 \cap K k_2 = 0$, $c_{11} = c_{22} = 0$, so necessarily $ba = 0$. This shows that $Sa \cap R = 0$. Now $e = k_2 a = e_{11} + k_1 k_2^{-1} e_{12}$ belongs to $Sa$, and $e = e^2$. It is easy to see that $eSe$ is a division ring (or equivalently, that $Se$ is a minimal left ideal of $S$), while $eSe \cap R = 0$. In particular, $eSe$ is not a quotient ring of $eSe \cap R$.

In view of Theorem 3, it is of interest to consider conditions which imply that the maximal right quotient ring is also a left quotient ring. The general question has been extensively treated by Utumi [1].

References

C. Faith and Y. Utumi

G. Findlay and J. Lambek

N. Jacobson

R. E. Johnson

R. E. Johnson and E. T. Wong

Y. Utumi