Let $R$ be an associative ring in which an identity element is not assumed. A right quotient ring of $R$ is an overring $S$ such that for each $a \in S$ there corresponds $r \in R$ such that $ar \in R$ and $ar \neq 0$. A theorem of R. E. Johnson [1] states that $R$ possesses a right quotient ring $S$ which is a (von Neumann) regular ring if and only if $R$ has vanishing right singular ideal. In this case $R$ possesses a unique (up to isomorphism over $R$) maximal right quotient ring $S$, and $S$ is regular and right self-injective (Johnson-Wong [1]). It is easy to see that $S$ is the injective hull of $R$, considering both rings as right $R$-modules in the natural way. Thus, each right ideal $I$ of $R$ has an injective hull $I_R$ contained in $S$. In this notation, $S = \hat{R}_R$, and we use $\hat{R}$ to denote the maximal right quotient ring of $R$ hereafter. By the results of Johnson [2], $I_R$ can be characterized in two ways:

(a) $I_R$ is the unique maximal essential extension of $I$ contained in the right $R$-module $\hat{R}$.

(b) $I_R$ is the principal right ideal of $\hat{R}$ generated by $I$.

Since $I_R$ is therefore a right ideal of $\hat{R}$, $\Delta = \text{Hom}_{\hat{R}} (I_R, I_R)$ is defined. Setting $\Gamma = \text{Hom}_R (I, I)$, one of our main results (Theorem 2) states that $\Gamma = \text{Hom}_R (I_R, I_R) = \Delta$. This means that $\Gamma$ has vanishing right singular ideal, and that $\Delta$ is the maximal right quotient ring of $\Gamma$.

Since $I_R$ is a principal right ideal in the regular ring $\hat{R}$, there exists an idempotent $e \in \hat{R}$ such that $I_R = e\hat{R}$. Then, of course, $\Delta \cong e\hat{R}e$, and it is natural to investigate the relationship between $e\hat{R}e$ and $K = e\hat{R}e \cap R$. In general, it is too much to hope that $e\hat{R}e = \hat{R}$, since it is possible that $K = 0$ for some nonzero $e \in \hat{R}$. Nevertheless, under the assumption that $\hat{R}$ is also a left quotient ring of $R$, or in case $e$ is a primitive idempotent satisfying $e\hat{R}e \cap R \neq 0$, we establish (Theorem 3) that $K$ has vanishing right singular ideal, and that $e\hat{R}e = \hat{R}$ (= the maximal right quotient ring of $K$.) In any case, for any nonzero idempotent $e \in \hat{R}$, $e\hat{R}e$ is the maximal right quotient ring of $e\hat{R}$. Since we are not restricting ourselves to rings with identity, we say...
that an arbitrary (right) module $M_R$ over a ring $R$ is injective in case it possesses the following property: if $A_R$ is any module, and if $B_R$ is any submodule, then any homomorphism $B_R \rightarrow M_R$ can be extended to a homomorphism $A_R \rightarrow M_R$. In case $R$ has an identity element, then a unital module $M_R$ is $\kappa$-injective in case it has the property above with $A_R$ ranging over all unital modules. It is easy to see that a unital module $M_R$ is injective if and only if it is $\kappa$-injective (cf. Faith-Utumi [1]). Baer's criterion (loc. cit.) states that a unital module $M_R$ is injective if and only if it has the property: If $f$ is any module homomorphism of a right ideal $I$ of $R$ into $M$, then there exists $m \in M$ such that $f(x) = mx \forall x \in I$. Call this latter property of a module $M_R$ Baer's condition. It is known (loc. cit.) that if $M_R$ is an arbitrary injective module, then it satisfies Baer's condition. Accordingly, if $S$ is any ring which is right self-injective, the identity map $x \rightarrow x$ can be performed by a left multiplication by an element $e \in S$ which is patently a left identity element of $S$. If $S$ is left-faithful, any left identity is two-sided. In particular, any semiprime right self-injective ring possesses an identity element. We use this fact below.

**Theorem 1.** If $S$ is semiprime and right self-injective, then for any idempotent $e \in S$, $eSe$ is semiprime and right self-injective.

**Proof.** Let $I$ be any right ideal of $eSe$ which is nilpotent of index 2. Then $(IS)^2 = (IS)(IS) = (I e S)(e I S) = [I(e Se)] IS \subseteq I^2 S = 0$. Thus, $IS$ is a nilpotent right ideal of $S$, whence $IS = 0$ and $I = 0$. Since $eSe$ does not contain nilpotent right ideals of index 2, it follows that $eSe$ is semiprime.

Now let $I$ be any right ideal of $eSe$, let $x = \sum_{i=1}^{n} x_i s_i$, $x_i \in I$, $s_i \in S$, $i = 1, \ldots, n$ be any element of $IS$. Let $f \in \text{Hom}_{eSe}(I, eSe)$, and let $T$ denote the set of all elements $\sum_{i=1}^{n} f(x_i) r_i \in \sum_{i=1}^{n} f(x_i) S$, $r_i \in S$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} x_i r_i = 0$. Clearly $T$ is a right ideal of $S$, and $T \subseteq \sum_{i=1}^{n} f(x_i) S \subseteq eS$. Now if $t = \sum_{i=1}^{n} f(x_i) r_i \in T$, then

$$te = \left[ \sum_{i=1}^{n} f(x_i) r_i \right] e = \sum_{i=1}^{n} [f(x_i) e] r_i = \sum_{i=1}^{n} f(x_i) (e r_i)$$

$$= \sum_{i=1}^{n} f(x_i e r_i) = f \left( \sum_{i=1}^{n} x_i r_i e \right) = f(0) = 0.$$

Thus $T^2 = (e T)^2 = 0$, and $T = 0$, since $S$ is semiprime. It follows that $x = \sum_{i=1}^{n} x_i s_i = 0$ implies that $\sum_{i=1}^{n} f(x_i) s_i = 0$, so that the correspondence

$$f': x \rightarrow \sum_{i=1}^{n} f(x_i) s_i,$$
defined for any \( x \in IS \), is an element of \( \text{Hom}_S(IS, S) \). Since \( S_S \) is injective, and \( S \) is semiprime, \( S \) has an identity element, so \( S_S \) satisfies Baer's condition. Accordingly, there exists \( m \in S \) such that \( f'(x) = m x \forall x \in IS \). In particular, if \( x \in I \), then \( x = xe \), so that \( f'(x) = f(x) e = f(x) \). Thus, \( f(x) = (eme) x \forall x \in I \). Since \( eme \in eSe \), this shows that \( (eSe)_S \) satisfies Baer's condition, and \( eSe \) is therefore right self-injective.

The proof of the next theorem needs some facts about essential and rational extensions, and the proofs of these may be found in Findlay-Lambek [1] and Johnson-Wong [1]. Recall that \( M_R \) is an essential extension of a submodule \( N_R \) in case each nonzero submodule of \( M_R \) has nonzero intersection with \( N_R \); we symbolize this by \( (M \triangledown N)_R \); \( r(M \triangledown N) \) denotes the right-left symmetry for a left module \( rM \). Then \( N_R \) is an essential submodule of \( M_R \). An essential left ideal \( I \) of \( R \) is a left ideal such that \( R \triangledown I \).

The singular submodule \( Z(M_R) \) is defined by:

\[
Z(M_R) = \{ x \in M \mid xf = 0 \text{ for some essential right ideal } I \text{ of } R \}.
\]

We let \( Z_r(R) \) denote \( Z(R_R) \); it is an ideal, called the right singular ideal of \( R \).

A module \( M_R \) is rational over a submodule \( N_R \) in case it has the following property: if \( P \) is any module satisfying \( M \supset P \supset N \), and if \( f \in \text{Hom}_R(P, M) \), then \( f = 0 \) if and only if \( f(N) = 0 \). We let \( (M \triangledown N)_R \) denote a rational extension \( M \) of \( N \). Any rational extension is essential; moreover:

If \( Z(N_R) = 0 \), then \( (M \triangledown N)_R \) if and only if \( (M \triangledown N)_R \).

We shall use the following characterization of rational extensions:

\( (M \triangledown N)_R \) if and only if for each pair \( x, y \in M \) with \( y \neq 0 \), there exist \( r \in R \) and an integer \( n \) such that

\[
xr + xn \in N \quad \text{and} \quad yr + yn \neq 0.
\]

We also need the following facts about quotient rings (Johnson [2]): Let \( R \) be such that \( Z_r(R) = 0 \), and adopt the notation of the introduction. For each right ideal \( I \) of \( R \), let \( \bar{I} = I R \cap R \). Then \( \bar{I} \) is the unique maximal essential extension of \( I_R \) contained in \( R \); \( I \) is a closed right ideal of \( R \) in case \( I = \bar{I} \). The totality \( C_r(R) \) of closed right ideals of \( R \) is a complete lattice, and \( C_r(\bar{R}) \) is isomorphic \( C_r(R) \) under the contraction map \( A \to A \cap R \). From what we already have said, \( C_r(\bar{R}) \) consists of the principal right ideals of \( \bar{R} \).

**Theorem 2.** Let \( R \) be a semiprime ring such that \( Z_r(R) = 0 \), let \( I \) be any right ideal of \( R \), let \( \bar{I} \) denote the principal right ideal of \( \bar{R} \) generated by \( I \), let \( \Gamma = \text{Hom}_R(I, I) \), and let \( \Delta = \text{Hom}_R(\bar{I}, \bar{I}) \). Then \( Z_r(\Gamma) = Z_r(\Delta) = 0 \), and \( \Delta = \bar{\Gamma} \) (the maximal right quotient ring of \( \Gamma \)).
Proof. Set $S = \mathcal{R}$. As remarked above, $\mathcal{I}_R$ is the injective hull of $I_R$ contained in $S_R$, and is the principal right ideal of $S$ generated by $I$. Since $S$ is regular, $\mathcal{I}_R = eS$, where $e = e^2 \in S$.

We first show that $\Delta = \text{Hom}_S(\mathcal{I}_R, \mathcal{I}_R)$ coincides with $\Omega = \text{Hom}_R(I, I)$. Clearly $\Omega \subseteq \Delta$. Conversely if $f \in \Omega$, and if $r \in \mathcal{I}_R \cap R = I$, then $f(r) = f(e)r$. If $x \in \mathcal{I}_R$, then $x_R = \{ t \in R \mid xt \in I \}$ is an essential right ideal of $R$. Now if $t \in x_R$, then

$$f(x)t = f(xt) = f(e)xt,$$

that is, $(f(x) - f(e)x)t = 0$. Thus, $[f(x) - f(e)x]x_R = 0$. Since $\mathcal{I}_R$ has zero singular submodule, we conclude that $f(x) = f(e)x\forall x \in \mathcal{I}_R$, and then clearly $f \in \Delta$. Thus establishes $\Omega = \Delta$.

Since $\mathcal{I}_R$ is the injective hull of $I_R$, it follows that each $\gamma \in \Gamma$ has an extension $\hat{\gamma} \in \mathcal{A} = \Omega$, and $\hat{\gamma}$ is unique, since $\mathcal{I}_R$ is rational over $I$. Clearly $\{ \gamma \in \Delta \mid \gamma \in \Gamma \}$ is a subring of $\Delta$ isomorphic to $\Gamma$ under $\gamma \mapsto \hat{\gamma}$. Henceforth, consider $\Gamma$ as a subring of $\Delta$.

Now $\Delta$ is isomorphic to the ring $eSe$. If $I_L$ denotes the totality of left multiplications $a_L$ of $I$ by elements $a \in I$,

$$a_L: x \rightarrow ax \quad x \in I,$$

then $I_L$ is a subring of $\Gamma$, and the natural isomorphism $\Delta \cong eSe$ maps $I_L$ onto $eIe$ and maps $\Gamma$ onto a subring $\Gamma_e$ of $eSe$. Since $\Gamma_e \cong eIe$, in order to show that $(eSe \cap \Gamma_e)$, it suffices to show that $(eSe \cap eIe)_e$.

Now let $0 \neq \delta \in eSe$. Since $(eSe \cap eIe)_e$, there exists $r \in R$ such that $0 \neq \delta r \in I$. Since $\delta r = \delta(\delta r)$, it follows that $\delta r \neq 0$. By the same reasoning, since $(eSe \cap eIe)_e$, there exists $s \in R$, $n \in Z$, such that $u = \delta(s+n) \in I$, and $w = \delta(r(s+n)) \neq 0$. Since $\delta r \in I$, it follows that $w \in I$. Since $R$ is semiprime, $R$ is left-faithful, hence $wR \neq 0$ and also $(wR)^2 \neq 0$. Therefore, one can choose $t \in R$ such that $w' = wt$ satisfies $w' = w' \neq 0$. Then $w' = \delta u'$, where $u' = ut \in I$ and $w' \in I$, and

$$0 \neq w'' = \delta u'',$

with $u'' = u'e \in eIe$ and $w'' \in eIe$. Thus, $(eSe \cap eIe)_e$ as asserted. Hence $eSe$ is a right quotient ring of $\Gamma_e$, and $\Delta$ is a right quotient ring of $\Gamma$. Now $S$ is regular (hence semiprime), so that $\Delta = eSe$ is right self-injective by Theorem 1. Thus, $\Delta$ is a maximal right quotient ring of $\Gamma$. Since $\Delta(\cong eSe)$ is regular, $Z(\Gamma) = Z(\Delta) = 0$.

Theorem 3. Let $R$ be any semiprime ring satisfying $Z(\Gamma) = 0$, and let $e$ be any idempotent in $S = \mathcal{R}$. Then:

1. $eSe = \mathcal{R}$, where $K = eRe$.

2. If $S$ is also a left quotient ring of $R$, then $eSe = \mathcal{R}$, where $K = eSe \cap R$, and $eSe$ is also a left quotient ring of $K$.  

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If \( e \) is a primitive idempotent, and if \( eS \cap R \neq 0 \), then \( eS \) is the right quotient field of \( K = eS \cap R \).

**Proof.** Let \( B = (eS \cap R) + (1 - e)S \cap R \). The lattice isomorphism \( C_{\nu}(R) \cong C_{\nu}(R) \) implies that

\[
[eS \cap R]_R \quad \text{(resp. \((1 - e)S \cap R]_R)),}
\]

and it follows that \((eS \cap B)_R\). For each \( x \in S \), \( x_B = \{ b \in B \mid xb \in B \} \) is a right ideal of \( B \), and \((R \cap x_B)_R\).

Now choose \( \delta \in eS \) and \( \delta \neq 0 \). Since \( \delta \) is an essential right ideal of \( R \), then \( \delta \delta_R \neq 0 \) (since \( Z(S) = Z(R) = 0 \)). By semiprimeness of \( R \), we see that \( \delta \delta_R \neq 0 \), so that \( \delta \delta_R \neq 0 \). Hence we can choose \( b \in \delta \subset B \) such that \( \delta \delta_R \). (Note that \( eb \in eS \cap R \).)

**Case (1).** Now \( \delta b \in R \), and \( b \in R \), and \( b \delta b = b(\delta b) = e(\delta b)b \). Thus \( 0 \neq b(eb) \in Re \) with \( eb \in eR \). This shows that \( eS \) is a right quotient ring of \( eR \).

**Case (2).** Since \( S \) is a regular ring which is both a right and left quotient ring of \( R \), necessarily \( Z(R) = 0 \). Then \( S \) is a rational extension of \( R \), and, moreover, \( S(eS) \) is a rational extension of \( R(eS \cap R) \).

Now the correspondence \( x \to x \delta b e \chi x \in S \) is an element \( f \in \text{Hom}_R(S, S) \), and \( f \neq 0 \) since \( e(\delta b e) = b \delta e \neq 0 \). It follows that \( f(S(eS \cap R)) \neq 0 \), that is, that \( (S(eS \cap R))b \delta e \neq 0 \) and \( (S(eS \cap R)) \delta b \neq 0 \). By the semiprimeness of \( R \), \( [(eS \cap R) \delta b]^2 \neq 0 \), and so \( b \delta (S(eS \cap R)) \). Hence choose \( u \in S(eS \cap R) \) such that \( \delta bu \neq 0 \). Since \( b, \delta b, u \in R \), then also \( \delta bu \), \( bu \in R \), and

\[
\delta bu = (eb)u(e) = e(\delta bu) \epsilon eS \cap R = K.
\]

Since \( eb \in eS \cap R \), and \( u \in S(eS \cap R) \), then \( k = ebu \in K \), so that \( 0 \neq k = bu \in K \), with \( k \in K \). Since \( \delta \) was an arbitrary nonzero element of \( eS \), this proves that \( eS \) is a right quotient ring of \( K \).

**Case (3).** \( eS \) is a division ring and \( S \) is a right vector space over \( eS \). Since \( S(eS \cap R) \neq 0 \), and since \( 0 \neq b \delta be \in eS \), it follows that \( (S(eS \cap R)) \delta be \neq 0 \), and the rest of the proof proceeds as in the proof of (2).

In all cases we have deduced that \( eS \) is a right quotient ring of \( K \) without resource to the fact that the right quotient ring \( S \) of \( R \) is maximal. Now in Case (2), \( Z(R) = 0 \) is a consequence of the fact that \( S \) is a regular ring which is a left quotient ring of \( R \). Hence, by symmetry, we conclude that \( eS \) is a left quotient ring of \( K \) in this case.

Since \( eS \) is regular along with \( S \), and since \( eS \) is right self-injective by Theorem 1, we conclude that \( eS = K \) in all cases, completing the proof.

**Remark.** It can be shown that \( K \) in (2) need not be semiprime.
We construct an example which shows that (2) and (3) of Theorem 3 fail under a weakening of the hypothesis.

Let $K$ be a right Ore domain which is not a left Ore domain, and let $k_1, k_2$ be nonzero elements of $K$ such that $Kk_1 \cap Kk_2 = 0$.

If $D$ denotes the right quotient division ring of $K$, then $S = D_2$, the full ring of all $2 \times 2$ matrices over $D$, is the classical, and maximal, right quotient ring of $R = K_2$. Let $\{e_{ij} | i, j = 1, 2, Z\}$ denote matrix units in $S$, let $a = k_1^{-1}e_{11} + k_2^{-1}e_{12}$, and suppose $b \in S$ is such that $ba \in R$. Then $b = \sum_{i,j=1}^{2} c_{ij} e_{ij}$, with $c_{ij} \in D$, $i, j = 1, 2$, and

$$ba = c_{11}k_1^{-1}e_{11} + c_{12}k_2^{-1}e_{12} + c_{21}k_1^{-1}e_{21} + c_{22}k_2^{-1}e_{22}.$$ 

Since $ba \in K_2 = R$, necessarily

$$c_{11}k_1^{-1}, c_{12}k_2^{-1}, c_{21}k_1^{-1}, c_{22}k_2^{-1} \subseteq K,$$

and then $c_{11}, c_{12} \in Kk_1 \cap Kk_2$. Since $Kk_1 \cap Kk_2 = 0$, $c_{11} = c_{22} = 0$, so necessarily $ba = 0$. This shows that $Sa \cap R = 0$. Now $e = k_2 a = e_{11} + k_1 k_2^{-1} e_{12}$ belongs to $Sa$, and $e = e^2$. It is easy to see that $eSe$ is a division ring (or equivalently, that $Se$ is a minimal left ideal of $S$), while $eSe \cap R = 0$. In particular, $eSe$ is not a quotient ring of $eSe \cap R$.

In view of Theorem 3, it is of interest to consider conditions which imply that the maximal right quotient ring is also a left quotient ring. The general question has been extensively treated by Utumi [1].

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