ON THE COMBINATORIAL SCHOENFLIES CONJECTURE

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Introduction. The combinatorial version of the Schoenflies conjecture in dimension $n$ states: A combinatorial $(n-1)$-sphere on a combinatorial $n$-sphere decomposes the latter into two combinatorial $n$-cells. The cases $n=1, 2$ are obvious, the case $n=3$ was solved by J. W. Alexander [1], see also W. Graeub [5], and E. Moise [8]. Nothing is known for $n>3$ (compare J. F. Hudson and E. C. Zeeman in [6, p. 729]). On the other hand the following form of the Hauptvermutung for spheres is proved: If a combinatorial $n$-dimensional manifold is homeomorphic an $n$-dimensional sphere, then it is a combinatorial $n$-sphere if $n \neq 4$. This was proved by S. Smale for $n \neq 4, 5, 7$ in [9], and improved by E. C. Zeeman to $n \neq 4$ (unpublished). I would like to thank Professor Zeeman for pointing out this result to me. Further a generalized Schoenflies theorem was proved by M. Brown in [3]. If this Hauptvermutung were true for all $n$, a simple induction on the dimension $n$ would prove the combinatorial Schoenflies conjecture for all $n$ (compare Theorem 6). In the following we show that either the combinatorial Schoenflies conjecture is true for all dimensions $n$, or it is false for all dimensions $n>3$.

Notation. We are dealing with finite simplicial complexes, hereafter called complexes. An $n$-complex is a complex such that each simplex is a face of an $n$-dimensional simplex of the complex. If $K, L$ are complexes then $K \cup L$, $K \cap L$, $K \cdot L$ denote the union, intersection, and the simplicial product of the complexes $K$ and $L$. A combinatorial $n$-cell is a complex which has a subdivision isomorphic a subdivision of an $n$-simplex. And a combinatorial $n$-sphere is a complex which has a subdivision isomorphic to a subdivision of the boundary of an $(n-1)$-simplex. The link $L(s, K)$ of the simplex $s$ of the complex $K$ is the subcomplex consisting of those simplexes $s'$ of $K$ with $s$ and $s'$ are faces of a simplex in $K$ and $s'$ has no vertices in common with $s$. An $n$-dimensional combinatorial manifold is an $n$-complex such that the link of each vertex is a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-cell. It is called closed, if its boundary is empty.

Theorem 1 (J. W. Alexander [2]). Let $S^n$ and $S^k$ be combinatorial spheres of dimensions $n$ and $k$, then the product $S^n \cdot S^k$ is a combinatorial $(n+k+1)$-sphere. Let $S^n$ be a combinatorial $n$-sphere and $E^n$ a com-
binatorial $k$-cell, then the product $S^n \cdot E^k$ is a combinatorial $(n+k+1)$-cell.

**Theorem 2** (J. W. Alexander [2]). Let $S^n$ be a combinatorial $n$-sphere with the decomposition $S^n = E^n_1 \cup E^n_2$, where $E^n_1$ and $E^n_2$ are $n$-complexes such that $E^n_1 \cap E^n_2 = S^{n-1}$ is a combinatorial $(n-1)$-sphere. If $E^n_1$ is a combinatorial $n$-cell, then $E^n_2$ is also a combinatorial $n$-cell.

**Theorem 3** (H. Kneser [7]). Let $M^n$, $n \geq 1$, be a closed connected $n$-dimensional combinatorial manifold. Suppose $H_1(M^n; \mathbb{Z}_2) = 0$ if $n \geq 2$, where $H_1(M^n; \mathbb{Z}_2)$ is the first homology group of $M^n$ with coefficients in $\mathbb{Z}_2$. Then any closed connected $(n-1)$-dimensional combinatorial manifold $M^{n-1}$ ($M^0 = S^0$) on $M^n$ determines a unique decomposition of $M^n$ into two $n$-complexes $K_1$ and $K_2$ with $M^n = K_1 \cup K_2$ and $K_1 \cap K_2 = M^{n-1}$.

**Theorem 4.** The combinatorial Schoenflies conjecture is true in dimension $n$ if and only if a combinatorial $n$-sphere $S^n$ on a combinatorial $(n+1)$-sphere $S^{n+1}$ decomposes the latter into two combinatorial manifolds $E_1^{n+1}$ and $E_2^{n+1}$ with $E_1^{n+1} \cup E_2^{n+1} = S^{n+1}$ and $E_1^{n+1} \cap E_2^{n+1} = S^n$.

**Proof.** The “if” part: Let $S^{n-1}$ be a combinatorial $(n-1)$-sphere on the combinatorial $n$-sphere $S^n$. From Theorem 3 we have the decomposition of $S^n$ into two $n$-complexes $E_1^n$ and $E_2^n$ with $E_1^n \cup E_2^n = S^n$ and $E_1^n \cap E_2^n = S^{n-1}$. Let $S^0 = \{x, y\}$ be a 0-sphere. We form the simplicial product $S^n \cdot S^0 = E_1^n \cdot S^0 \cup E_2^n \cdot S^0$. Of course $E_1^n \cdot S^0 \cap E_2^n \cdot S^0 = S^{n-1} \cdot S^0$. But $S^{n-1} \cdot S^0$ is now a combinatorial $n$-sphere on the combinatorial $(n+1)$-sphere $S^n \cdot S^0$. And $E_1^n \cdot S^0$ and $E_2^n \cdot S^0$ are then by hypothesis combinatorial manifolds. We compute the following links:

$$L(x, E_1^n \cdot S^0) = E_1^n$$

and

$$L(x, E_2^n \cdot S^0) = E_2^n.$$

$E_1^n$ and $E_2^n$ must be $n$-cells.

The “only if” part: From Theorem 3 we have that $S^{n+1}$ is decomposed into two $(n+1)$-complexes $E_1^{n+1}$ and $E_2^{n+1}$ with $S^{n+1} = E_1^{n+1} \cup E_2^{n+1}$ and $E_1^{n+1} \cap E_2^{n+1} = S^n$. To show that these $(n+1)$-complexes are combinatorial manifolds, we have only to check that the links of their vertices are either combinatorial $n$-spheres or combinatorial $n$-cells. For a vertex $e$ not in $S^n$ we have $L(e, E_1^{n+1}) = L(e, S^{n+1})$, $i = 1, 2$, and this link is therefore a combinatorial $n$-sphere. If the vertex $e$ lies on $S^n$ then the combinatorial $(n-1)$-sphere $L(e, S^n)$ decomposes the combinatorial $n$-sphere $L(e, S^{n+1})$ by hypothesis into two combinatorial $n$-cells, which lie in $E_1^{n+1}$ and $E_2^{n+1}$. These $n$-cells are the links of $e$ in $E_1^{n+1}$ and $E_2^{n+1}$.

**Corollary 1.** If the combinatorial Schoenflies conjecture is true in dimension $n$, then it is also true in dimension $n-1$, and therefore in all dimensions up to $n$. 

Theorem 5 (M. Brown [3] and [4]). Let $S^n$ be a combinatorial $n$-sphere and $S^{n-1}$ a combinatorial $(n-1)$-sphere on $S^n$. Then $S^{n-1}$ decomposes $S^n$ into two complexes $E_1^n$ and $E_2^n$ with $S^n = E_1^n \cup E_2^n$, $E_1^n \cap E_2^n = S^{n-1}$, and $E_1^n, E_2^n$ are topological $n$-cells, i.e. $E_1^n$ and $E_2^n$ are homeomorphic to an $n$-simplex.

Proof. $S^{n-1}$ is “bi-collared” in $S^n$, and the generalized Schoenflies theorem of [3] can be applied (see [4]). By the combinatorial Hauptvermutung for $n$-spheres we mean the statement, that if an $n$-dimensional combinatorial manifold is homeomorphic to an $n$-sphere then it is a combinatorial $n$-sphere.

Theorem 6. If the combinatorial Schoenflies conjecture is true in dimension $n-1$, and if the combinatorial Hauptvermutung for $n$-spheres is true, then the combinatorial Schoenflies conjecture is true in dimension $n$.

Proof. Suppose $S^{n-1}$ is a combinatorial $(n-1)$-sphere on the combinatorial $n$-sphere $S^n$. From Theorem 4 we have the decomposition $S^n = E_1^n \cup E_2^n$, where $E_1^n$ and $E_2^n$ are $n$-dimensional combinatorial manifolds, and $E_1^n \cap E_2^n = S^{n-1}$. By Theorem 5 we know that $E_1^n$ and $E_2^n$ are topological $n$-cells. We consider the $n$-dimensional combinatorial manifold $\tilde{S}^n = E_1^n \cup S^{n-1} \cdot E^0$, where $E^0 = \{x\}$ is a 0-cell. Since $E_1^n$ and $S^{n-1} \cdot E^0$ are topological $n$-cells, $\tilde{S}^n$ is homeomorphic to the $n$-sphere, and by hypothesis $\tilde{S}^n$ is a combinatorial $n$-sphere. Since $S^{n-1} \cdot E^0$ is a combinatorial $n$-cell, we can apply Theorem 2 and conclude that $E_1^n$ is also a combinatorial $n$-cell. And by the same argument $E_2^n$ is a combinatorial $n$-cell too.

Theorem 7 (S. Smale [9] and E. C. Zeeman). The combinatorial Hauptvermutung for $n$-spheres is true for $n \geq 4$.

Corollary 2. Either the combinatorial Schoenflies conjecture is true for all dimensions $n$, or it is false for all dimensions $n > 3$.

Corollary 3. If the combinatorial Schoenflies conjecture can be proved for one dimension $n_0 > 3$, then it is true for all dimensions $n$. If a counterexample to the combinatorial Schoenflies conjecture can be found in one dimension $n_0 > 3$, then the conjecture is false for all dimensions $n > 3$, and subsequently the combinatorial Hauptvermutung for 4-spheres would be false.

References