The purpose of this note is to prove an existence theorem for immersions of piecewise linear manifolds in Euclidean space. A more comprehensive theory of piecewise linear immersions has been worked out by Haefliger and Poenaru [1].

All maps, manifolds, microbundles, etc. are piecewise linear unless the contrary is explicitly indicated.

Let M be a manifold without boundary, of dimension n. Denote the tangent microbundle of M by \( \tau_M \), and the trivial microbundle over M of (fibre) dimension \( k \) by \( \varepsilon^k \). Let

\[
\nu: M \to E \to \mathbb{R}^{n+k}
\]

be a microbundle of dimension k such that E is a manifold. An immersion of M in \( \mathbb{R}^{n+k} \) is a locally one-one map \( f: M \to \mathbb{R}^{n+k} \).

I say \( f \) has a normal bundle of type \( \nu \) if there is an immersion \( g: E \to \mathbb{R}^{n+k} \) such that \( gi = f \). (It is unknown whether \( f \) necessarily has a normal bundle, or whether all normal bundles of \( f \) are of the same type.)

The converse of the following theorem is trivial.

**Theorem.** Assume that if \( k = 0 \), then M has no compact component. There exists an immersion of M in \( \mathbb{R}^{n+k} \) having a normal bundle of type \( \nu \) if there exists an isomorphism

\[
\phi: \tau_M \oplus \nu \to \varepsilon^{n+k}
\]

**Proof.** We may assume that \( i(M) \) is a deformation retract of the total space E of \( \nu \). By Milnor [3], \( \tau_E | i(M) \) is isomorphic to \( \tau_M \oplus \nu \); it follows from the existence of \( \phi \) that \( \tau_E \) is trivial. According to [3]...
there is a parallelizable differential structure \( \alpha \) on \( E \) compatible with the piecewise linear structure. Let \( h: E_a \rightarrow \mathbb{R}^{n+k} \) be a differentiable immersion, which exists by Hirsch [2] or Poenaru [4]. (If \( k = 0 \), the assumption that \( M \) has no compact component is used here.) Approximate \( h \) by a piecewise linear immersion \( g: E \rightarrow \mathbb{R}^{n+k} \), using the theory of \( C^1 \) complexes of Whitehead [5]. Clearly \( gi: M \rightarrow \mathbb{R}^{n+k} \) is an immersion having a normal bundle of type \( \nu \).

**Remarks.** (1) The assumption that \( M \) is unbounded is unnecessary, since a bounded manifold can be embedded in its interior. However, \( \tau_M \) must be redefined if \( M \) has a boundary.

(2) It is not hard to define the concepts of “immersion plus normal bundle”—essentially an immersion of \( E \) and of a “regular homotopy” of these; one can then prove a uniqueness theorem.

**Bibliography**


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