LOWER BOUNDS FOR SOLUTIONS OF HYPERBOLIC INEQUALITIES

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1. Introduction. Let $D$ denote a bounded domain in $E^n$ and $I$ the interval $1 \leq t < \infty$. Let $L$ be the second-order hyperbolic operator

\begin{equation}
L = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)
\end{equation}

defined on $R=D \times I$. Introducing the norms

\begin{align*}
\|u(t)\|_0 &= \int_D u^2 \, dx, \\
\|u(t)\|_1 &= \int_D \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 \right] \, dx,
\end{align*}

for functions $u$ in $C^2(R)$, Protter [4] investigated the asymptotic behavior of solutions of inequalities of the form

\begin{equation}
\|u(t)\|_0^2 \leq \alpha(t) \|u(t)\|_1.
\end{equation}

If $\Gamma$ is the boundary of $D$, he found that any solution of (1.2) which satisfies the conditions

\begin{equation*}
u = 0 \quad \text{on} \quad \Gamma \times I,
\end{equation*}

\begin{equation*}
\lim_{t \to \infty} t^\alpha \|u(t)\|_1 = 0 \quad \text{for all} \quad \alpha > 0,
\end{equation*}

must vanish identically, provided that

\begin{equation}
\phi(t) = O(t^{-1}), \quad \frac{\partial a_{ij}}{\partial t} = O(t^{-1}).
\end{equation}

Conditions for other types of asymptotic behavior have also been studied by Protter [5].

It is the purpose of this paper to find sufficient conditions for the existence of lower bounds of the form

\begin{equation*}
\|u(t)\|_1 \geq C \|u(t_0)\|_1 [K(t)]^{-1}, \quad t \geq t_0 \geq 1,
\end{equation*}

where $C$ is a positive constant and $K$ is a differentiable function satisfying

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In particular, it will be shown that in the case \( K(t) = t^a \), a lower bound exists under conditions somewhat weaker than (1.3). The results will also be extended to symmetric hyperbolic operators.

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2. Second-order hyperbolic inequalities. Let \( L \) be the operator defined by (1.1). We assume \( a_{ij} = a_{ji} \in C^1(\mathbb{R}) \), and suppose that there are positive constants \( m \) and \( M \) such that

\[
\sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq M \sum_{i=1}^{n} \xi_i^2.
\]

For functions \( u \in C^2(\mathbb{R}) \) we introduce the norm

\[
\|u(t)\|_1 = \int_{\Omega} \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right] dx,
\]

which is equivalent to the norm \( \|u(t)\|_1 \). If \( u = 0 \) on \( \Gamma \times I \), it is easily seen that

\[
\frac{d}{dt} \|u(t)\|_1^2 = 2 \int_{\Omega} \frac{\partial u}{\partial t} L u dx + \int_{\Omega} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial t} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.
\]

Hence for any function \( K \) satisfying (1.4), we have the identity

\[
\frac{d}{dt} [K(t) \|u(t)\|_1^2] = 2K(t)K'(t)\|u(t)\|^2 + 2K^2(t) \int_{\Omega} \frac{\partial u}{\partial t} L u dx + K^2(t) \int_{\Omega} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial t} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.
\]

(2.1)

Assume \( u \) is a solution of

\[
\|Lu(t)\|_0 \leq \phi(t) \|u(t)\|_1,
\]

such that \( u = 0 \) on \( \Gamma \times I \), and let \( \psi \) be a function satisfying

\[
\int_{\Omega} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial t} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \leq 2\phi(t) \|u(t)\|_1^2.
\]

(2.3)

Applying Schwarz's inequality and (2.2) to the second term on the right-hand side of (2.1), and applying (2.3) to the third term of the same expression, we find that
where we have set \( f = \phi + \psi \). It follows that

\[
\frac{d}{dt} \left[ \frac{K^2(t)}{K(t)} \| u(t) \|^2 \right] \geq 2K^2(t) \| u(t) \|^2 \left[ \frac{K'(t)}{K(t)} - f(t) \right],
\]

if \( \| u(t) \| \neq 0 \).

**Theorem.** Let \( u \) be a solution of (2.2) such that \( u = 0 \) on \( \Gamma \times I \). If \( \| u(t_0) \| \neq 0 \) and either

(i) \( (K'/K)^{1/p-1} f \in L_p(1, \infty) \) for some \( p \), \( 1 \leq p < \infty \),

or

(ii) \( Kf'/K' \in L_\infty(1, \infty) \) and \( \| Kf'/K' \|_\infty \leq 1 \),

then there exists a positive constant \( C \) such that

\[
(2.5) \quad \| u(t) \| \geq C \| u(t_0) \| (K(t))^{-1}, \quad t \geq t_0 \geq 1.
\]

**Proof.** We first assume that \( \| u(t) \| \neq 0 \) for \( t \geq t_0 \). Integrating (2.4) between \( t_0 \) and \( t \) we obtain

\[
(2.6) \quad \log \frac{K(t)}{K(t_0)} \| u(t) \| \geq \log \frac{K(t)}{K(t_0)} - \int_{t_0}^{t} f ds.
\]

In case (i), Hölder's inequality implies that

\[
\left| \int_{t_0}^{t} f ds \right| \leq \left[ \int_{t_0}^{t} \left( \frac{K'}{K} \right)^{-1/q} f^{p/q} ds \right] \left[ \int_{t_0}^{t} \frac{K'}{K} ds \right]^{1/q}
\]

\[
\leq N \left[ \log \frac{K(t)}{K(t_0)} \right]^{1/q},
\]

where \( N \) is a constant and \( 1/p + 1/q = 1 \). Hence, since \( \lim_{t \to \infty} K(t) = \infty \), we see that the right-hand side of (2.6) is bounded below, and (2.5) follows. Under case (ii), the right-hand side of (2.6) is easily seen to be non-negative.

To prove that the assumption \( \| u(t) \| \neq 0 \) is valid for all \( t \geq t_0 \), we suppose the contrary. Let \( t_1 > t_0 \) be the least value of \( t \) for which \( \| u(t) \| = 0 \). Then from the preceding result we find that (2.5) holds for \( t_0 \leq t < t_1 \). By the continuity of the norm, we must have \( \| u(t_1) \| \neq 0 \). This completes the proof of the theorem.
If \( K(t) = t^a, \alpha > 0 \), conditions (i) and (ii) become: either \( t^{1-1/p} \in L_p(1, \infty) \) for some \( p, 1 \leq p < \infty \), or \( tf \in L_\infty(1, \infty) \) and \( \| f \|_\infty \leq \alpha \), which include Protter's conditions (1.3). For \( K(t) = e^{at}, \alpha > 0 \), the conditions for the corresponding lower bound are: \( f \in L_p(1, \infty) \) for some \( p, 1 \leq p < \infty \), or \( f \in L_\infty(1, \infty) \) and \( \| f \|_\infty \leq \alpha \). These include the conditions obtained by Protter in [5], and are comparable to those found by Protter [3], Cohen and Lees [2] and Agmon and Nirenberg [1] for solutions of parabolic inequalities.

3. Symmetric hyperbolic inequalities. Let \( u \) be a \( k \)-component vector function in \( C^1(R) \) and denote the components of \( u \) by \( u^i, j = 1, 2, \cdots, k \). For such functions we define

\[
Lu = A_0 \frac{\partial u}{\partial t} + \sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i},
\]

where the \( A_i, i = 0, 1, \cdots, n \), are symmetric \( k \)-by-\( k \) matrices with elements in \( C^1(R) \), and \( A_0 \) is positive definite. We take as norms the quantities

\[
||u(t)||_0^2 = \int_D (u, u) dx,
\]

\[
||u(t)||^2 = \int_D (A_0 u, u) dx,
\]

with

\[
(u, v) = \sum_{j=1}^{k} u^j v^j.
\]

Since \( A_0 \) is symmetric, we have

\[
\frac{d}{dt} ||u(t)||^2 = \int_D \left( 2 A_0 \frac{\partial u}{\partial t} + \frac{\partial A_0}{\partial t} u, u \right) dx
\]

\[
= \int_D \left( 2 Lu - 2 \sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i} + \frac{\partial A_0}{\partial t} u, u \right) dx.
\]

Similarly, it follows that

\[
\int_D \left( A_i \frac{\partial u}{\partial x_i}, u \right) dx = -\frac{1}{2} \int_D \left( \frac{\partial A_i}{\partial x_i}, u \right) dx,
\]

for functions \( u \) which vanish on the boundary \( \Gamma \times I \). Thus, defining
we find that

\[
\frac{d}{dt} ||u(t)||^2 = \int_D (2Lu + Bu, u)dx.
\]

Suppose \(u\) is a solution of (2.2) and \(u\) vanishes on \(\Gamma \times I\). Let \(\psi\) be a function satisfying

\[
\left| \int_D (Bu, u)dx \right| \leq 2\psi(t)||u(t)||^2.
\]

Then the identity (3.1) implies that \(u\) satisfies the inequality (2.4), so the theorem of §2 is also valid in the present case.

**Bibliography**


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