INNER AND OUTER FUNCTIONS ON RIEMANN SURFACES

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1. Introduction. In this paper we generalize to Riemann surfaces the factorization theory for functions in the Hardy classes, H^p, on the unit disk.

Let R be a region on a Riemann surface with boundary Γ consisting of a finite number of simple closed analytic curves such that $R \cup \Gamma$ is compact and R lies on one side of Γ. For $1 \leqq p < \infty$ $H^p(R)$ is the class of functions F analytic on R such that $|F|^p$ has a harmonic majorant. $H^\infty(R)$ is the class of bounded analytic functions on R. The above classes are the usual generalizations of the Hardy classes on the disk (cf. [**4**], [**5**], and [**6**]). However, to obtain a factorization of these functions which closely parallels the factorization on the disk we are led to more general classes of functions. To this end, we say that a (multiple-valued) analytic function F on R is *multiplicative* if $|F|$ is single-valued and define $MH^p(R)$, $1 \leqq p < \infty$, to be the class of multiplicative analytic functions F on R such that $|F|^p$ has a harmonic majorant. Also, we define $MH^\infty(R)$ to be the class of bounded multiplicative functions on R.

Let $d\mu$ be the harmonic measure on Γ with respect to some fixed point $t_0 \in R$. If $F \in MH^p$ then $|F|$ has nontangential boundary values $|F^*|$ a.e. $[d\mu]$ on Γ. Moreover, $|F^*| \in L^p(\Gamma, d\mu)$ and $\log|F^*| \in L^1(\Gamma, d\mu)$ if $F \not\equiv 0$. These facts follow easily from the corresponding results on the disk. (Cf. [**10**, p. 496].)

We say $F \in MH^p(R)$ is an *outer function* if

$$\log|F(t_0)| = \int_\Gamma \log|F^*|\, d\mu.$$

$\Phi \in MH^\infty(R)$ is an *inner function* if $|\Phi^*| = 1$ a.e. on Γ. A nonvanishing inner function is said to be a *singular function*. An inner function, B, is said to be a *Blaschke product* if

$$|B(t)| = \exp\left[-\sum_k p_k G(t, t_k)\right]$$

for all $t \in R$, where G is the Green's function for R, $\{t_k\}$ is a sequence of points on R, and $\{p_k\}$ is a sequence of non-negative integers. When

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R is the unit disk, the definitions above are equivalent to the classical ones.

THEOREM 1. *Let $F \in MH^p(R)$. Then $|F| = |B||S||F_1|$ where B is a Blaschke product, S is a singular function, and F_1 is an outer function in $MH^p(R)$. These factors are unique up to multiplicative constants of modulus one.*

Since $H^p(R) \subset MH^p(R)$, this theorem subsumes a factorization of functions in $H^p(R)$. However, the factors of a single-valued function in $H^p(R)$ need not be single-valued.

Theorem 1 is stated in terms of the moduli of functions since there is no natural way to define the product of multiplicative functions; that is, the product would depend on the branches chosen.

In our proof we shall make use of the factorization on the disk. A convenient reference for this is [3].

It should be remarked that results related to ours are in [7] for R an annulus and in [2] for the general case.

2. **Proof of Theorem 1.** Let $K = \{z \cup z| < 1\}$ and $T: K \rightarrow R$ be a universal covering map of R. Let $Q = \{q\}$ be the group of fractional linear transformations such that $T \circ q = T$. A function f, analytic on K, is said to be *modulus invariant* if $|f \circ q| = |f|$ for all $q \in Q$. It is easy to see that f is modulus invariant on K if, and only if, $f \circ T^{-1}$ is multiplicative on R. Also, for f modulus invariant, $f \in H^p(K)$ if, and only if, $f \circ T^{-1} \in MH^p(R)$. For $F \in MH^p(R)$ and $f \circ T^{-1} = F$ let $f = bsf_1$ where b is a Blaschke product, s is a singular function, and f_1 is an outer function in $H^p(K)$. Then f is modulus invariant and by Lemma 4.6 in [10] bs and f_1 are also modulus invariant.

LEMMA. *b is modulus invariant.*

PROOF. First observe that if z is a zero of f then $q(z)$ is a zero of f with the same multiplicity for each $q \in Q$. Thus $b \circ q/b \in H^\infty(K)$ for each $q \in Q$ since b and $b \circ q$ have the same zeros with the same multiplicities. Then for all $q \in Q$ $b/b \circ q = (b \circ q^{-1} \circ q)/(b \circ q) \in H^\infty(K)$. It follows that $|b| = |b \circ q|$ for all $q \in Q$.

Let Δ be the fundamental domain of Q and E the union of the free sides of Δ; that is, $E = \bar{\Delta} \cap \{|z| = 1\}$. Then the harmonic measure $d\mu$ corresponds to the measure

$$dm(\theta) = \frac{1}{2\pi}\left[\sum_{q \in Q} \frac{1 - |q(z_0)|^2}{|e^{i\theta} - q(z_0)|^2}\right] d\theta$$

on E where $z_0 \in \Delta$ and $T(z_0) = t_0$. (Cf. [9, pp. 526, 529].) Moreover, $dm(\theta)$ and $d\theta$ are mutually absolutely continuous.

Let $B=b\circ T^{-1}$, $S=s\circ T^{-1}$, and $F_1=f_1\circ T^{-1}$. Then $|F| = |B|\,|S|\,|F_1|$. Since b is an inner function and s is a singular function, it is immediate by virtue of the relation between $d\theta$ and $dm(\theta)$ that B is an inner function and S is a singular function. (Cf. [**10**, Lemma 4.4].) It remains to show that B is a Blaschke product and F_1 is an outer function.

We shall consider F_1 first. Since $dm(\theta)$ corresponds to $d\mu$, it follows that

$$\int_\Gamma \log|F_1^*|\,d\mu = \int_E \log|f_1^*|\,dm(\theta).$$

Now since the set $\bigcup_{q\in Q} q(E)$ is of full measure on $|z|=1$ (see [**9**, p. 525]) and $|f_1^*\circ q| = |f_1^*|$ a.e. on $|z|=1$, a change of variable for the integral on the right (see [**9**, pp. 526–528]) yields

$$\begin{aligned}\int_\Gamma \log|F_1^*|\,d\mu &= \int_E \log|f_1^*|\,dm(\theta)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|f_1^*|\,\frac{1-|z_0|^2}{|e^{i\theta}-z_0|}\,d\theta\\ &= \log|f_1(z_0)| = \log|F_1(t_0)|.\end{aligned}$$

Therefore F_1 is an outer function.

Next we show that B is a Blaschke product. Let $\{t_k\}$ be the set of zeros of F and p_k be the multiplicity of the zero at t_k. For each t_k let z_k be a point in K such that $T(z_k)=t_k$. Then for $q\in Q$, b has a zero at $q(z_k)$ of multiplicity p_k. All the zeros of b occur in this way. Thus,

$$|b(z)| = \prod_k\prod_{q\in Q}\left|\frac{q(z_k)-z}{1-\bar q(z_k)z}\right|^{p_k}.$$

Now for $z, z'\in K$

$$G(T(z),T(z')) = \sum_{q\in Q}\log\left|\frac{1-\bar q(z')z}{z-q(z')}\right|.$$

(See [**9**, p. 529].) Hence for $t=T(z)$

$$|B(t)| = |b(z)| = \prod_k \exp[-p_kG(t,t_k)] = \exp\left[-\sum_k p_kG(t,t_k)\right]$$

Thus B is a Blaschke product and $F=|B|\,|S|\,|F_1|$ is the desired factorization. Since b, s, and f_1 are unique up to multiplicative constants of modulus one, the same is true for B, S, and F_1. This completes the proof of Theorem 1.

3. Closed invariant subspaces of $H^2(R)$. Let $A(R)$ be the class of (single-valued) functions continuous on $\overline{R}$ and analytic on R. A closed subspace V of $H^2(R)$ is said to be *invariant* if $FV \subset V$ for all $F \in A(R)$. For Φ an inner function on R let $V(\Phi) = \{F \in H^2(R) \mid |F|^2/|\Phi|^2$ has a harmonic majorant on $R\}$. In [10, Theorem 8.11] it is shown that $V(\Phi)$ is a closed invariant subspace of $H^2(R)$. The following theorem, which reduces to a well-known result of Beurling's [1, Theorem IV, p. 253] for the case $R=K$, is proved in [10, Theorem 2].

THEOREM 2. *If V is a closed invariant subspace of $H^2(R)$ then there is an inner function Φ such that $V=V(\Phi)$.*

By Theorem 1 we have for $F \in H^p(R)$, $|F| = |\Phi|\ |F_1|$ where Φ is an inner function. We call Φ an *inner factor of F*.

THEOREM 3. *Let $F \in H^2(R)$ and $V[F]$ be the smallest closed invariant subspace of $H^2(R)$ which contains F. Then $V[F] = V(\Phi)$ where Φ is an inner factor of F.*

PROOF. Clearly $V(\Phi) \supset V[F]$. By Theorem 2, $V[F] = V(\Phi_0)$ for some inner function Φ_0. Let $H \in V(\Phi)$. We must show $H \in V(\Phi_0)$. For $f = F \circ T$ and $h = H \circ T$ let $f = \phi f_1$ and $h = \psi h_1$ be the inner-outer factorizations of f and h respectively such that $\Phi = \phi \circ T^{-1}$. Then $\Psi = \psi \circ T^{-1}$ is an inner factor of H. Let ϕ_0 be a modulus invariant inner function such that $\Phi_0 = \phi_0 \circ T^{-1}$. Since $F \in V(\Phi_0)$, $|F|^2/|\Phi_0|^2$ has a harmonic majorant; and it follows that $f/\phi_0 \in H^2(K)$. This implies ϕ/ϕ_0 is an inner function. By a similar argument ψ/ϕ is an inner function. Thus $\psi/\phi_0 = (\psi/\phi)(\phi/\phi_0)$ is an inner function. Hence $|\Psi|/|\Phi_0|$ is bounded. This implies $H \in V(\Phi_0)$.

COROLLARY 1. *Let $F \in H^2(R)$. Then $V[F] = H^2(R)$ if, and only if, F is an outer function.* (*Cf.* [7, *Theorem* 1, p. 128].)

The following result was proved by D. Sarason ([7, pp. 112, 128] and [8, Theorem 4, p. 596]). We offer a different proof.

COROLLARY 2. *Let $R = \{z \mid r < |z| < 1\}$ and suppose $F \in H^2(R)$. Then $V[F] = H^2(R)$ if, and only if, for $0 \leqq \delta \leqq 1$*

$$(*) \qquad \int_0^{2\pi} \log |F(r^\delta e^{it})|\, dt = (1-\delta)\int_0^{2\pi} \log |F^*(e^{it})|\, dt + \delta \int_0^{2\pi} \log |F^*(re^{it})|\, dt.$$

PROOF. By virtue of Corollary 1 it is sufficient to show that F satisfies (*) if, and only if, F is an outer function. Suppose F is an outer function, then

$$\begin{aligned}\int_0^{2\pi} \log|F(r^\delta e^{it})|\,dt &= \int_0^{2\pi} \frac{1}{2\pi}\int_{\partial R} \log|F^*(w)|\, \frac{\partial G(w, r^\delta e^{it})}{\partial n}\,|dw|\,dt \\ &= \int_{|w|=1} \log|F^*(w)|\, \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial G(w, r^\delta e^{it})}{\partial n}\,dt\,|dw| \\ &+ \int_{|w|=r} \log|F^*(w)|\, \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial G(w, r^\delta e^{it})}{\partial n}\,dt\,|dw| \\ &= I_1(\delta) + I_2(\delta).\end{aligned}$$

Now $\partial G(e^{i\theta}, r^\delta e^{it})/\partial n = \partial G(e^{-it}, r^\delta e^{-i\theta})/\partial n$. Thus for $|w| = 1$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial G(w, r^\delta e^{it})}{\partial n}\,dt &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial G(e^{it}, r^\delta \bar{w})}{\partial n}\,dt \\ &= \text{harmonic measure of } \{\,|z| = 1\} \text{ at } r^\delta\bar{w} \\ &= 1 - (\log r^\delta/\log r) = 1 - \delta.\end{aligned}$$

Thus $I_1(\delta) = (1-\delta)\int_0^{2\pi} \log|F^*(e^{it})|\,dt$. A similar argument shows $I_2(\delta) = \delta\int_0^{2\pi} \log|F^*(re^{it})|\,dt$. Hence F satisfies (*).

To complete the proof we show that if F is not an outer function then (*) is not satisfied. Let F_1 be an outer factor of F. Then $|F| < |F_1|$ on R and $|F| = |F_1|$ a.e. on ∂R. We have shown that F_1 satisfies (*). Thus (*) does not hold for F.

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