A NOTE ON THE HAUSDORFF MOMENT PROBLEM

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In [1, pp. 630–635], J. H. Wells presented a solution of the Hausdorff moment problem for the case of a quasicontinuous mass function. The purpose of this note is to extend that result to include Riemann-integrable mass functions.

If \( \{d_n\} \) is a number sequence, let \( A_{np} = \sum_{n=p}^{\infty} d_n \), \( n \geq p, p = 0, 1, 2, \ldots \). We observe that [1, p. 634, Theorem 2.4(ii)(b)] may be stated as follows:

If \( \epsilon > 0 \), there is a finite collection \( C \) of nonoverlapping subsegments \( (u, v) \) of the segment \( (0, 1) \) such that \( \sum C (v - u) = 1 \) and if \( u < y < z < v \), then there is a positive integer \( N \) such that if \( n > N \),

\[ \left| \sum_{n=p}^{\infty} A_{np} + \sum_{n=p}^{\infty} A_{np} \right| \leq \epsilon. \]

The arguments used to establish [1, p. 634, Theorem 2.4] and the associated theorems and lemmas [1, pp. 630–633] are readily modified to supply a proof of the following theorem.

**Theorem.** If \( \{d_n\} \) is a number sequence, the following two statements are equivalent:

(i) There is a function \( g \) Riemann-integrable on \( [0, 1] \) such that \( d_n = \int_{[0,1]} t^n dt, n = 0, 1, 2, \ldots \);

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(ii) (a) there is a number $M$ such that $|\sum_{p=0}^{k} A_{np}| < M$, $0 \leq k \leq n$, $n = 0, 1, 2, \ldots$, and

(b) if $\varepsilon > 0$ and $0 < \delta < 1$, there is a finite collection $C$ of nonoverlapping subsegments $(u, v)$ of the segment $(0, 1)$ such that $\sum \alpha(v-u) > 1 - \delta$ and if $u < y < z < v$, then there is a positive integer $N$ such that if $n > N$,

$$|\sum_{np \leq p} A_{np} + \sum_{np < np} A_{np}| < \varepsilon.$$ 

The crux of the matter lies in the observation that [1, p. 633, Lemma 2.3] holds if the mass function is Riemann-integrable on $[0, 1]$, and in noticing the following Ascoli-type result (compare with [1, p. 630, Theorem 2.1]):

LEMMA. Suppose $\{f_n\}$ is a uniformly bounded infinite sequence of real functions from $[0, 1]$ and if $\varepsilon > 0$ and $0 < \delta < 1$, there is a finite collection $C$ of nonoverlapping subsegments $(u, v)$ of the segment $(0, 1)$ such that $\sum \alpha(v-u) > 1 - \delta$ and if $u < y < z < v$, then there is a positive integer $N$ such that if $n > N$,

$$|f_n(y) - f_n(x)| < \varepsilon,$$

and $\{g_n\}$ is an infinite subsequence of $\{f_n\}$ which converges at each point of a countable set which is dense in $[0, 1]$. If, for each $x$ in $[0, 1]$, $h(x)$ is a cluster point of $\{g_n(x)\}$, then on $[0, 1]$ $h$ is Riemann-integrable and $\{g_n\}$ converges almost everywhere to $h$.

REFERENCE


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