ITERATED $w^*$-SEQUENTIAL CLOSURE OF A BANACH SPACE IN ITS SECOND CONJUGATE

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1. If $X$ is a real Banach space and $S$ is a subspace of the second conjugate space $X^{**}$, let $K(S)$ be the $w^*$-sequential closure of $S$ in $X^{**}$; thus if $F \in X^{**}$, then $F \in K(S)$ if and only if $F$ is the $w^*$-limit of a sequence in $S$. If $J$ is the canonical mapping from $X$ into $X^{**}$, then $K(JX)$ is norm-closed in $X^{**}$ [3]. In the present paper it is shown that $K(K(JX))$ can fail to be norm-closed in $X^{**}$, thus answering a question raised in [4]. The proof is a modification of the proof in [4] that $K(K(JP))$ can fail to be norm-closed in $X^{**}$ for $P$ a cone in $X$.

2. If $[a; b]$ is a finite real interval, let $B[a; b]$ be the Banach space of all bounded real functions $z$ on $[a; b]$ with $\|z\| = \sup \{ |z(t)| : a \leq t \leq b \}$. If $A$ is a subset of $B[a; b]$, let $L(A)$ be the set of all $z \in B[a; b]$ such that $z$ is the pointwise limit of a bounded sequence in $A$. If $X$ is a closed subspace of the Banach space $C$ of all continuous functions on $[a; b]$, and if $\{x_n\}$ is a bounded sequence in $X$ which is pointwise convergent to a function $z \in L(X)$, Lebesgue's bounded convergence theorem [2, p. 110] shows that $\{Jx_n\}$ is $w^*$-convergent to an element $F \in K(JX)$ which depends only on $z$. The correspondence $z \rightarrow F$ thus obtained is an isometric isomorphism from $L(X)$ onto $K(JX)$, and for simplicity of notation we shall henceforth identify $L(X)$ and $K(JX)$. In like manner, $L(L(X))$ and $K(K(JX))$ are isometrically isomorphic and will be identified with each other.

The main result will be shown to follow from the following theorem.

**Theorem 1.** For each number $c \geq 1$ there exists a Banach space $X$ with the property that there is an $x_0 \in K(K(JX))$ such that $\|x_0\| = 1$, but if $\{y^k\}$ is a sequence in $K(JX)$ which is $w^*$-convergent to $x_0$, then $\lim \inf_k \|y^k\| \geq c$.

**Proof.** If $s$, $\alpha$, and $\beta$ are positive numbers such that $0 \leq s - \alpha - \beta$ and $s + \alpha + \beta \leq 3$, let $f(s; \alpha, \beta)$ be the continuous real function on $[0; 3]$ whose values at the points $0, s - \alpha - \beta, s - \alpha, s + \alpha, s + \alpha + \beta$, and $3$ are $0, 0, 1, 1, 0$, and $0$ respectively and which is linear on each of the five subintervals so determined.

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If \( u, v, i, j \) are in the set \( \omega \) of all positive integers, and \( i < 2^u \), let \( s_{ui} = 2^{-u}i \) and \( t_{ej} = 2 - 2^{-e}(1 + 2^{-i}) \). By induction on \( p \), it is possible to choose irrational positive numbers \( c_{piq} \), where \( p, i, q \in \omega \) and \( i < 2^p \), such that

(i) \( c_{piq} = 2^{p-1}c_{piq} \);

(ii) \( 2 \leq 2 + s_{pi} \pm 2c_{pi} \leq 3 \);

(iii) if \( i \) is odd, then \( f(2 + s_{pi}; c_{pi}, c_{pi}) \) is linear on the interval \([2 + s_{pi} - 2c_{pi}; 2 + s_{pi} + 2c_{pi}]\) if \( p' < p \) and \( j < 2^{p'} \);

(iv) \( f(2 + s_{pi}; c_{pi}, c_{pi}) \) vanishes at the points \( 2 + s_{p'i} \pm 2c_{p'i} \) if \( p' \leq p \) and \( j < 2^{p'} \).

Let \( X \) be the closed subspace of \( e[0;3] \) generated by \( \{x_{pq}: p, q \in \omega \} \), where

\[
x_{pq} = \sum_{i=1}^{2^{p-1}} f(s_{pi}; 5 \cdot 2^{-p-i}, 2^{-p-i})
\]

\[+ c \sum_{v=1}^{p} \sum_{j=p}^{p+q-1} f(t_{vj}; 2^{-v-j-2}, 2^{-v-j-2})
\]

\[+ \sum_{i=1}^{2^{p-1}} f(2 + s_{pi}; c_{pi}, c_{pi}). \]

For each subset \( S \) of \([0; 3]\) let \( \chi(S) \) be the characteristic function of \( S \) relative to \([0; 3]\). For each fixed \( p \in \omega \), the sequence \( \{x_{pq}\}_{q \in \omega} \) is pointwise convergent to the function

\[
x_p = \chi(\{s_{pi}: i < 2^p\} \cup \{2 + s_{pi}: i < 2^p\})
\]

\[+ c \chi(\{t_{pj}: v \leq p \leq j\}), \]

and the sequence \( \{x_p\}_{p \in \omega} \) is pointwise convergent to the function

\[
x_0 = \chi(\{s_{pi}: p \in \omega, i < 2^p\} \cup \{2 + s_{pi}: p \in \omega, i < 2^p\}). \]

It is easily verified that the norm of each \( x_{pq} \) and each \( x_p \) is \( c \), whereas \( \|x_0\| = 1 \). Thus \( \{x_p: p \in \omega\} \subseteq K(JX) \), and \( x_0 \in K(K(JX)) \).

Now let \( \{y^k\}_{k \in \omega} \) be an arbitrary sequence in \( K(JX) \) which is \( w^* \)-convergent to \( x_0 \); we must show that \( \liminf_{k} \|y^k\| \geq c \). Each \( y^k \) can be regarded as the pointwise limit of a bounded sequence \( \{y^h^k\}_{h \in \omega} \) in \( X \). Without changing \( \liminf_{k} \|y^k\| \), we can assume that \( y^h(1/2) = 1 \) for each \( h \). Since \( \{x_{pq}: p, q \in \omega\} \) generates \( X \), each \( y^h \) can be taken in the form

\[
y^h = \sum_{p, q \in \omega} a_{pq} x_{pq}, \]

for some constants \( a_{pq} \).
where for each pair \((h, k)\) only a finite number of the coefficients \(a_{pq}^h\) are different from zero. Moreover, by a proof in [3, pp. 333–334], for each \(h\) the sequence \(\{y^h\}_k\in\omega\) can be chosen so that \(\lim_k \|y^h\| = \|y^h\|\).

Let \(\varepsilon > 0\) be given. For each \(H \in \omega\) let

\[
(2.5) \quad S_H = \bigcap \{t \in [0; 1]: |y^h(t)| < \varepsilon \text{ and } |y^h(2 + t)| < \varepsilon\}. 
\]

Since \(\bigcup_{H \in \omega} S_H\) contains all but a countable number of the points of \([0; 1]\) and is hence of the second category, there exists an integer \(H_0 \in \omega\) such that \(S_{H_0}\) is dense in a closed interval \(I\). There exist \(p_0 \in \omega\) and an odd integer \(i_0\), with \(1 \leq i_0 < 2p_0\), such that the interval \(I_0 = [2^{-p_0}; 2^{-p_0} + 2^{-p_0} - 1]\) is contained in \(I\).

Now \(x_0(t_v) = 0\) for each \(v \in \omega\). Hence there exists \(H_1 \supseteq H_0\) such that

\[
|y^h(t_v)| < \varepsilon p_0^{-1} \quad \text{for all } h \geq H_1 \text{ and } v < p_0. 
\]

Observe that \(y^{hk}(t_{pp} - 2^{-2p - e_2} - y^{hk}(t_{pp} - 2^{-2p - e_1}) = a_{pq}^{hk}\) in every case. Thus \(a_{pq}^{hk} = \lim_k a_{pq}^k\) exists for all \(h, p, q\), and \(\lim_k a_{pq}^k = 0\) for every pair \((p, q)\). Since \(i_0\) is odd, there are only a finite number \(N\) of pairs \((p, q)\) such that \(p < p_0\) and \(x_{pq}\) assumes nonzero values on \(I_0\). Hence there exists \(H_2 \supseteq H_1\) such that \(a_{pq}^h < \varepsilon N^{-1}\) for each \(h \geq H_2\) and each pair \((p, q)\) such that \(x_{pq}\) assumes nonzero values on \(I_0\) and \(p < p_0\).

Now fix an integer \(H_2 \geq H_2\). Since \(y^h\) is a Baire function of the first class, \(y^h\) must have a point of continuity in the interval \(G = \{2 + t: t \in I\}\); then since \(|y^h(s)| < \varepsilon\) for all \(s\) in a dense subset of \(G\), it follows that there is a closed subinterval \(G_0\) of \(G\) such that \(|y^h(s)| < \varepsilon\) for all \(s \in G_0\). There exist \(p_1 > p_0\) and an odd integer \(i_1\), with \(1 \leq i_1 < 2p_1\), such that \(G_1 = [2 + s_{pi_1}; 2 + s_{pi_1} + 2c_{pi_1} + 2c_{pi_1}] \subseteq G_0\). By the conditions on the numbers \(c_{piq}\), each \(x_{pq}\) with \(p < p_1\) is linear on \(G_1\), and each \(x_{pq}\) with \(p \geq p_1\) vanishes at the endpoints of \(G_1\). Hence for each \(k \in \omega\),

\[
y^{hk}(2 + s_{pi_1}) = \sum_{p \geq p_1; q \in \omega} a_{pq}^{hk} + \frac{1}{2} y^{hk}(2 + s_{pi_1} - 2c_{pi_1}) + y^{hk}(2 + s_{pi_1} + 2c_{pi_1}),
\]

and so there exists \(K_0 \in \omega\) such that

\[
(2.7) \quad \sum_{p \geq p_1; q \in \omega} a_{pq}^{hk} < 4\varepsilon \quad \text{for all } k \geq K_0.
\]

Let \(z_1 = 2^{-p_0} i_0 + 2^{-p_1 - 2}\) and let \(z_2\) be that integral multiple of \(2^{-p_1 - 2}\) which is closest to \(2^{-p_0} (i_0 + 3^{-1})\). From (2.1) it can be veri-
fied that there is an interval \( M_2 \) centered at \( z_2 \) such that if \( t \in M_3 \), then \( x_{pq}(t) = 0 \) for all \( p, q \) such that \( p_0 \leq p \leq p_1 + 1, \ q \in \omega \). There is an interval \( M_1 \) centered at \( z_1 \) such that for all \( t \in M_1 \),

\[
\begin{cases} 
1 & \text{if } p_0 \leq p \leq p_1 \text{ and } p + q \leq p_1 - 1, \\
0 & \text{if } p_0 \leq p \leq p_1 + 1 \text{ and } p + q \geq p_1.
\end{cases}
\]

(2.8) \[ x_{pq}(t) \]

It may be assumed that \( M_1 \) and \( M_2 \) have the same length. Since \( M_1 \subseteq I \) and \( h \geq H_0 \), there must be a subinterval \( M_3 \) of \( M_1 \) such that \( |y^h(t)| < 2\varepsilon \) for all \( t \in M_3 \). Since the interval \( M'_3 = \{ t \in M_3 : t - (z_2 - z_1) \in M_3 \} \) is also contained in \( I \), there exists \( s \in M'_3 \) such that \( |y^s(z)| < \varepsilon \). If \( p \geq p_1 + 2 \), the function \( x_{pq} \) is periodic with period \( 2^{-p} \) on \( I_0 \); since \( z_2 - z_1 \) is an integral multiple of \( 2^{-p} \), it follows that \( x_{pq}(z) = x_{pq}(z - z_2 + z_1) \) whenever \( p \geq p_1 + 2 \).

Since \( h \geq H_2 \), there is an integer \( K_1 \geq K_2 \) such that if \( k \geq K_1 \) and \( t \in I_0 \),

(2.9) \[ \sum_{p < p_1, q \in \omega} a_{pq} x_{pq}(t) < \varepsilon. \]

Further, there is a \( K_3 \geq K_1 \) such that if \( k \geq K_3 \), then \( |y^h(z)| < \varepsilon \) and \( |y^{hk}(z - z_2 + z_1)| < 2\varepsilon \). Consequently, if \( k \geq K_3 \),

\[
\begin{align*}
&\sum_{p < p_1, q \in \omega} a_{pq} x_{pq}(t) \\
&= \left| y^h (z - z_2 + z_1) - y^h (z) \\
&\quad - \sum_{p < p_1, q \in \omega} a_{pq} x_{pq}(z - z_2 + z_1) \\
&\quad + \sum_{p < p_1, q \in \omega} a_{pq} x_{pq}(z) \right| < 5\varepsilon.
\end{align*}
\]

(2.10) \[ \sum_{p < p_1, q \in \omega} a_{pq} x_{pq}(t) \]

Since \( h \geq H_1 \) there exists \( K_3 \geq K_2 \) such that if \( k \geq K_3 \),

(2.11) \[ \sum_{p < p_1, q \in \omega} a_{pq} x_{pq}(t) < \varepsilon. \]

Since \( y^h(1/2) = 1 \), there exists \( K_4 \geq K_3 \) such that \( k \geq K_4 \) implies

(2.12) \[ \sum_{p, q \in \omega} a_{pq} x_{pq}(t) - 1 \]

Now \( y^{hk}(t_{p_0, p_1-1}) = c \sum_{p_0 \leq p \leq p_1-1, p \in \omega} a_{pq} x_{pq}(t) \), and hence if \( k \geq K_4 \) it follows from (2.7), (2.10), (2.11), and (2.12) that
\[
\left| y^{hk}(t_{p_0, p_1 - 1}) \right|
\]

(2.13) \[
\sum_{p, q \in \omega} a_{pq}^k - \sum_{p < p_0, q \in \omega} a_{pq}^k + \sum_{p < p_0 \leq p_1 - 1; q < p_1 - 1} a_{pq}^k - \sum_{p > p_1; q \in \omega} a_{pq}^k
\]

\[
> (1 - \varepsilon) - \varepsilon - 5\varepsilon - 4\varepsilon = 1 - 11\varepsilon.
\]

Thus \(\|y^{hk}\| > c(1 - 11\varepsilon)\) for all \(k \geq K_4\), and hence \(\|y^h\| = \lim_k \|y^{hk}\| \geq c(1 - 11\varepsilon)\).

We have shown that for every \(\varepsilon > 0\) there is an \(H_2\) such that \(\|y^h\| \geq c(1 - 11\varepsilon)\) for all \(h \geq H_2\). It follows that \(\lim \inf_{\sigma} \|y^h\| \geq c\).

**Theorem 2.** There exists a real Banach space \(X\) such that \(K(K(JX))\) is not norm-closed in \(X^{**}\).

**Proof.** For each \(n \in \omega\) there is, by Theorem 1, a Banach space \(X_n\) such that \(K_n(K_n(J_nX_n))\) contains an element \(x_n\) with \(\|x_n\| = 1\), such that if \(\{y_n^h\}_{h \in \omega}\) is a sequence in \(K_n(J_nX_n)\) which is \(\omega^*\)-convergent to \(x_n\), then \(\lim \inf_h \|y_n^h\| \geq c\). Here \(J_n\), for example, is the canonical mapping from \(X_n\) into \(X_n^{**}\), and \(K_n(S)\) is the \(\omega^*\)-sequential closure of \(S\) in \(X_n^{**}\) for each \(S \subseteq X_n^{**}\).

Let \(X\) be the Banach space \(P_{\omega^*(\omega)}X_n\) as defined in \([1, \text{p.} 31]\). Then \(X^*\) is isometrically isomorphic with \(P_{\omega^*(\omega)}X^*_n\), and \(X^{**}\) with \(P_{\omega^*(\omega)}X^{**}_n\); we shall make the obvious identifications. Now \(x_0 = (n^{-1}x_{on})_{n \in \omega} \in X^{**}\), and indeed \(x_0 \in \operatorname{Cl}(K(K(JX)))\), since \(x_0\) is the limit in norm of the sequence \(\{z_i\}_{i \in \omega}\), where \(z_i = (1^{-1}x_{01}, 2^{-1}x_{02}, \ldots, i^{-1}x_{0i}, 0, \ldots) \in K(K(JX))\) for each \(i \in \omega\).

If \(x_0 \in K(K(JX))\), then \(x_0\) would be the \(\omega^*\)-limit of a sequence \(\{w_i\}_{i \in \omega} \subseteq (K(JX))\). For each \(n\), \(n^{-1}x_{on}\) would be the \(\omega^*\)-limit of the sequence \(\{w_i^h\}_{i \in \omega} \subseteq K_n(J_nX_n)\); it would then follow that \(\lim \inf_h \|w_i^h\| \geq n^{-1}n^2 = n\) for each \(n \in \omega\), contradicting the fact that a \(\omega^*\)-convergent sequence must be bounded in norm. Consequently \(x_0 \in \operatorname{Cl}(K(K(JX)))\), and the proof is complete.

**References**


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