CONTRACTIBLE COMPLEXES IN $S^n$

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1. Introduction. By a pseudo $n$-cell is meant a contractible compact combinatorial $n$-manifold with boundary. Poenaru [9] and Mazur [7] gave the first examples of pseudo 4-cells which are not topological 4-cells, but whose products with the unit interval are topologically 5-cells. Newman [8] defines a 2-complex $P$ such that $\pi_1(P) \neq 1$, while $H_1(P, Z) = 0 = H_2(P, Z)$. Curtis [4] making use of this 2-complex has shown that, for each $n \geq 4$, there exists a pseudo $n$-cell which is not a topological $n$-cell because its boundary fails to be simply connected. Curtis [4] also shows that the cartesian product of a pseudo $n$-cell and an interval is the topological $(n+1)$-cell, $I^{n+1}$ if $n \geq 5$.

Curtis [5] making use of Mazur's peculiar embedding of the dunce hat in $S^4$ [7], [13] gives an example of a contractible 2-complex $K$ embedded as a subcomplex of a combinatorial triangulation of $S^4$ such that $\pi_1(S^4 - K) \neq 1$. The purpose of this paper is to show that for $n \geq 4$ there exists a contractible $(n-2)$-complex $K^{n-2}$ combinatorially embedded in $S^n$ such that $\pi_1(S^n - K^{n-2}) \neq 1$. The regular neighborhood $N^n = N(K^{n-2})$ of $K^{n-2}$ in $S^n$ is also a pseudo $n$-cell which fails to be a topological $n$-cell and its product with the unit interval $I$ is shown to be a combinatorial $(n+1)$-cell, rather than just merely topological. In addition, each $N^n$ ($n \geq 5$) gives examples of combinatorial $n$-manifolds with boundary which are not topologically $I^n$ but can be expressed as the union of two combinatorial $n$-balls whose intersection is also a combinatorial $n$-ball.

2. Definitions. We will use the terminology of [12], [13]. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. We will use $\approx$ to denote combinatorial equivalence. If the complex $K$ collapses to the complex $L$, this will be denoted $K \sim L$.

Let $f: X \to Y$ be continuous. The identification space $Y_f$ derived from $(X \times [0, 1]) \cup Y$ by identifying each point $(x, 1)$ with the point

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$f(x)$ in $Y$ and using the identification topology is called the mapping cylinder of $f$.

3. **Preliminaries.** The following two lemmas are well known and elementary, hence no proof will be included.

**Lemma 1.** If $C$ is a $k$-complex embedded as a finite subcomplex of a combinatorial $n$-sphere $S^n$ and $M$ is a regular neighborhood of $C$ in $S$, with $C \subset \text{int} M$, then there is a combinatorial map $\phi : \text{Bd} M \rightarrow C$ such that $M$ is combinatorially equivalent to $I \times \text{Bd} M \cup _{\phi} C$, the mapping cylinder of $\phi$.

**Lemma 2.** Suppose $C$ is a $k$-complex embedded as a finite subcomplex of a combinatorial $n$-sphere $S^n$ and $N$ is any regular neighborhood of $C$ in $S^n$, such that $C \subset \text{int} N$; then $\pi_1(N - C) = \pi_1(\text{Bd} N)$.

The topological dunce hat $D$ is obtained from a triangle $abc$ say, by identifying all three sides $ab = bc = ac$.

**Theorem 1.** There exist two combinatorially inequivalent embeddings $D_1$, $D_2$ of the dunce hat $D$ in $S^4$, such that the regular neighborhood $N_1$ of $D_1$ is combinatorially $I^4$, while $N_2$ the regular neighborhood of $D_2$ is not topologically $I^4$. Moreover, $\pi_1(\text{Bd} N_2) \neq 1$, $\pi_1(N_2 - D_2) \neq 1$, but $\pi_1(S^4 - D_2) = 1$.

**Proof of Theorem 1.** Let $D_1$ be any combinatorial embedding of $D$ in $S^4 \subset S^4$. Then $N_1 \sim N_1$, the regular neighborhood of $D$ in $S^4$, and since $N_1 \approx I^4$, $N_1 \approx I^4$.

For $D_2$ we will use Mazur's embedding of $D$ in $S^4$ (as in Theorem 5 [13]). Since $N_2 \approx W^4$ (also Theorem 5 [13]) and $\pi_1(\text{Bd} W^4) \neq 1$ (see [7]) we have that $N_2 \neq I^4$. The fact that $\pi_1(N_1 - D_2) \neq 1$ follows from Lemma 2. We see that $\pi_1(S^4 - D_2) = 1$ by considering Mazur's embedding of $D$ in $S^4$. That is $D \subset W^4 \subset 2W^4 = S^4$. Since $S^4 - D_2 \approx W^4 \cup W^4 - D_2 \approx W^4 \cup (\text{Bd} W^4 \times [0, 1])$ (using Lemma 1), we see that $S^4 - D_2$ is of the same homotopy type as $W^4$ and $\pi_1(S^4 - D_2) = 1$.

To see that these two embeddings are combinatorially inequivalent, suppose there exists a p.w.l. homeomorphism taking $S^4 \subset S^4$ carrying $D_1$ onto $D_2$. Let $a_1$, $a_2$ be the points of $D_1$, $D_2$ respectively, which correspond to the point $a(=b=c)$ in $D$. Then by subdividing the triangulation of $S^4$ so that $h$ is simplicial, we get that $h$ carries $\text{st}(a_1, S^4)$ onto $\text{st}(a_2, S^4)$, each combinatorial 4-balls. Also $h$ carries $\text{lk}(a_1, D_1) \subset \text{lk}(a_2, S^4) \approx S^3$ onto $\text{lk}(a_2, D_2) \subset \text{lk}(a_2, S^4) \approx S^3$. This leads to a contradiction, since there exists no homeomorphism of $S^4$ onto $S^4$ carrying $\text{lk}(a_1, D_1)$ as in $\text{lk}(a_1, S^4)$ onto $\text{lk}(a_2, D_2)$ as in $\text{lk}(a_2, S^4)$. See Figures 5 and 8 of [13].
Theorem 2. There exists a contractible 2-complex $K$ and two inequivalent embeddings $K_1, K_2$ of $K$ in $S^4$ so that the regular neighborhood $N_1$ of $K_1$ is a combinatorial 4-ball, while $\pi_1(S^4 - K_2) \neq 1$.

Remark. Since $N_1 \approx I^4$, $K_1$ is cellular in $S^4$ and hence $S^4 - K_1 = E^4$ and $\pi_1(S^4 - K_1) = 1$. Also it will follow from a later result, which does not use the particular construction of the embedding of $K_2$ in $S^4$, that if $N_2$ is the regular neighborhood of $K_2$ in $S^4$ then $\pi_1(\text{Bd} N_2) \neq 1$ and hence $N_2 \neq I^4$.

Proof of Theorem 2. $K$ will be the union of two disjoint copies of the dunce hat $D$ joined together by a polyhedral segment intersecting each in $a (= b = c)$. $K_1$ will be the embedding of $K$ in $S^3 \subset S^4$ and $N_1 \approx I^4$ as in Theorem 1.

To get $K_2$, we will use Curtis's modification [5]. Let us again consider $S^4$ as $2W^4$ (Mazur's pseudo 4-cell). We have a $D'$ and $D''$ (copies of $D$) in each copy of $W^4$. Since $S^4 - (D' + D'') \cong (W^4 - D') \cup (W^4 - D'') \cong (\text{Bd} W^4 \times [0, 1]) \cup (\text{Bd} W^4 \times [0, 1])$ and $\pi_1(\text{Bd} W^4) \neq 1$, we have $\pi_1(S^4 - (D' + D'')) \neq 1$. Let $A$ be a polyhedral arc in $S^4$ such that $A \cap D' = a'$, $A \cap D'' = a''$ (where $a'$, $a''$ correspond to $a (= b = c)$ in $D$) and $A \cap \text{Bd} W^4 = \{p\}$. Such an $A$ can easily be gotten because of the particular embedding of $D'$, $D''$ in each copy of $W^4$. Then $K_2 = D' \cup A \cup D''$ will be an embedding of $K$ in $S^4$ such that $\pi_1(S^4 - K_2) \neq 1$.

Finally, it is clear that the embeddings of $K_1$ and $K_2$ in $S^4$ are inequivalent since the fundamental groups of their complements are different.

Theorem 3. If $N_2$ is the regular neighborhood of $K_2$ in $S^4$ then $N_2 \times I \approx I^5$.

Proof of Theorem 3. Since $K_2 \approx D \cup A \cup D$, two disjoint copies of $D$ joined together by a polyhedra arc intersecting each $D$ in the point $a$ and $D \times I \setminus \{a\}$ (Theorem 1 [13]), it follows easily that $K_2 \times I \setminus 0$. Hence $N_2 \times I \setminus K_2 \times I \setminus 0$ and this implies that $N_2 \times I$ is a combinatorial 5-ball (Corollary 1 [12]).

Theorem 4. Suppose $K$ is a contractible 2-complex such that $K \times I \setminus 0$ and $K$ is embedded in the interior of a contractible 4-manifold with boundary $W^4 \subset E^4$ such that $\pi_1(W^4 - K) \neq 1$. Then given any combinatorial triangulation $T$ of $E^4$ which contains $K$ as a subcomplex, there exists no 3-manifold (with or without boundary) in $E^4$ which is a subcomplex of $T$ containing $K$.

Remark. Mazur's embedding of $D$ in $S^4$ is such a contractible 2-complex. It follows from the theorem that even though $D$ can be em-
bedded in $E^4$, for this particular embedding it lies in no 3-manifold in $E^4$.

**Proof of Theorem 4.** Suppose there exists such a 3-manifold $M^3$, that is $K \subset M^3 \subset T$. Then for some subdivision of $T$ and hence of $M^3$, say $\hat{T}$, we would have $N(K, M^3) \subset \text{int } W^4$, where $N(K, M^3)$ denotes the simplicial neighborhood of $K$ in $M^3$ under the second barycentric subdivision of $\hat{T}(M^3)$. Also let us suppose that $\hat{T}$ is so fine that $N(N(K, M^3), \hat{T}) \subset \text{int } W^4$. Now if $N(K, M^3) = I^3$, then $N(N(K, M^3), \hat{T}) = I^4 \subset \text{int } W^4$. We then could use $\text{Bd } I^4 = S^3$ to shrink nontrivial curves of $W^4 - K$ missing $K$. (Also see Theorem 6 of [13].) Therefore, $N(K, M^3) \neq I^3$. However, $N(K, M^3) \times I \times K \times I \times 0$ and this implies that $N \times I = I^4$ which in turn implies $N = I^4([1], [2])$ which contradicts the above. This contradiction arose by assuming there existed an $M^3$ with $K \subset M^3 \subset T$.

4. **Contractible complexes in $S^n$.** If $K$ is a $k$-complex of a combinatorial $n$-sphere $S^n$, we will use $N(K, S^n)$ to denote the canonical regular neighborhood of $K$ under the second barycentric subdivision of $S^n$. $\Sigma K$ and $CK$ will denote the suspension of $K$ and cone over $K$ respectively. Also, we will write $\Sigma K = C^+K \cup C^-K$ with $C^+K \cap C^-K = K$, where in letting $p$ and $q$ denote the “top” and “bottom” points of $\Sigma K$ used in getting the suspension of $K$, we have that $C^+K$ is the cone over $K$ in $\Sigma K$ from $p$ and $C^-K$ is the cone over $K$ in $\Sigma K$ from $q$.

**Lemma 3.** Suppose $\hat{K}$ is a $k$-complex in $S^n$ such that $N(\hat{K}, S^n) \approx I^n$ and $B^n$ is a combinatorial $n$-ball in $S^n$ such that $N\left(\hat{K}, S^n\right) \subset \text{int } B^*$. If $\Sigma \hat{K} = K$ is considered as embedded in $S^{n+1} = B^{n+1} \cup (\text{Bd } B^{n+1})$, where $B^{n+1} = \Sigma B^n$, then $N(K, S^{n+1}) \approx I^{n+1}$ and $\pi_1(S^{n+1} - K) = 1$.

**Proof of Lemma 3.** $\Sigma [N(\hat{K}, S^n)]$ is a regular neighborhood of $K$ in $S^{n+1}$. That is, $\Sigma [N(\hat{K}, S^n)] \setminus \Sigma \hat{K} = K$ since $N(\hat{K}, S^n) \setminus \hat{K}$ and it is an $n$-manifold with boundary since $\Sigma I^n \approx I^{n+1}$. Hence $I^{n+1} \approx \Sigma [N(\hat{K}, S^n)] \approx N(K, S^{n+1})$ (Theorem 23 [12]). It follows that $\pi_1(S^{n+1} - K) = 1$ since $K$ is cellular in $S^{n+1}$ (the decreasing sequence of $(n+1)$-cells are the canonical regular neighborhoods of $K$ under increasingly higher order barycentric subdivisions of $S^{n+1}$). That is $S^{n+1} - K = E^{n+1} [2]$. 

**Lemma 4.** Suppose $\hat{K}$ is a $k$-complex in $S^n$ ($n \geq 3$) such that $\pi_1(S^n - \hat{K}) \neq 1$ and $B^n$ is a combinatorial $n$-ball in $S^n$ such that $\hat{K} \subset \text{int } B^n$. Then if $\Sigma \hat{K} = K$ is considered as embedded in $S^{n+1}$ as in Lemma 3, then $\pi_1(S^{n+1} - K) \neq 1$.

**Proof of Lemma 4.** Since $\pi_1(S^n - \hat{K}) \neq 1$, we have that $\pi_1(B^n - \hat{K}) \neq 1$. Also $\Sigma B^n - \Sigma \hat{K} = B^{n+1} - K \approx (B^n - \hat{K}) \times (-1, 1)$. Hence
The claim is that $\pi_1(S^{n+1} - k) \neq 1$. Suppose otherwise. Let $J$ be any polyhedral simple closed curve in $B^{n+1} - K$ which is nontrivial in $B^{n+1} - K$. Suppose $J$ bounds a polyhedral singular disk $D$ in $S^{n+1} - K$. Let $p, q$ be the suspension points of $\Sigma B^n$ and $r$ the vertex point in $C(Bd(\Sigma B^n))$. Since $n + 1 \geq 4$, we can adjust $D$ to a singular disk $D'$ (keeping $J$ fixed) so that $D' \cap (\text{polyhedral arc } pq) = \emptyset$. But then $D'$ can be retracted onto a singular disk $D''$ bounded by $J$ in $B^{n+1} - K$ by projecting the part of $D'$ not in $S^{n+1}$ from $r$ onto $Bd B^{n+1} - \{p + q\}$. This leads to a contradiction that $\pi_1(B^{n+1} - K) \neq 1$, therefore $\pi_1(S^{n+1} - K) \neq 1$.

**Lemma 5.** If $K$ is a $k$-complex in $S^n$ and $\pi_1(S^n - K) \neq 1$, denoting $N(K, S^n)$ by $N$, then $N \neq I^n$, $\pi_1(N - K) = \pi_1(Bd N) \neq 1$ and $\pi_1(\text{Cl}(S^n - N)) \neq 1$.

**Proof of Lemma 5.** If $N = I^n$ then $K$ is cellular in $S^n$ and this would imply that $\pi_1(S^n - K) = 1$, contradicting the hypothesis of the lemma. Also, $S^n - K = (N - K) \cup \text{Cl}(S^n - N) = ([0, 1] \times Bd N) \cup \text{Cl}(S^n - N)$ (by Lemma 1). Hence $S^n - K$ is homotopically equivalent to $\text{Cl}(S^n - N)$. Therefore $\pi_1(\text{Cl}(S^n - N)) \neq 1$.

Suppose $\pi_1(Bd N) = 1$. Since $S^n = N \cup \text{Cl}(S^n - N)$ and $N \cap \text{Cl}(S^n - N) = Bd N$, if $\pi_1(Bd N) = 1$, then using van Kampen's theorem we get that $\pi_1(S^n)$ is the free product $\pi_1(N) * \pi_1(S^n - N)$, which would not be trivial (Corollary 6.4.5, p. 244, [6]). Therefore, $\pi_1(Bd N) \neq 1$ and by Lemma 2 $\pi_1(\text{Bd N}) = \pi_1(N - K) \neq 1$.

**Lemma 6.** Suppose $K$ is a $k$-complex in $S^n$ such that $K \times I \neq 0$. Let $K = \Sigma K$ be $p.w.l.$ embedded in $S^{n+1}$ (not necessarily as in Lemma 3), then $N(K, S^{n+1}) \times I = I^{n+1}$.

**Proof of Lemma 6.** First we note that if $L$ is a subcomplex of $K$ such that $K \setminus L, K = \Sigma \hat{K}$ and $L = \Sigma \hat{L}$ then $K \setminus L$. This follows by induction on the number of simplexes of $K - L$. Next we observe that if $K$ is a complex such that $K \setminus 0$ then $K = \Sigma \hat{K} \setminus 0$. This follows since $K \setminus \{v\}$ (v some vertex of $K$) and by the above remark $K = \Sigma \hat{K} \setminus \Sigma v \setminus v$. Finally, if $\hat{K}$ is a complex such that $\hat{K} \setminus I \neq 0$ and if $K = \Sigma \hat{K}$, then $K \setminus I \neq 0$. This follows since $\Sigma \hat{K} \setminus I \setminus (\hat{K} \setminus I) \setminus I$ by the second remark.

Therefore since $\hat{K} \setminus I \neq 0$, we have that $K \setminus I \neq 0$. Hence, $N(K, S^{n+1}) \setminus I \setminus K \setminus I \setminus I = I^{n+1}$.

**Theorem 5.** For $n \geq 4$ there exists a contractible $(n - 2)$-complex $P$ and two inequivalent embeddings $P_1, P_2$ of $P$ in $S^n$ such that the regular neighborhood $N_1$ of $P_1$ is a combinatorial $n$-ball and $\pi_1(S^n - P_1) = 1$. However, $\pi_1(S^n - P_2) \neq 1$ and if $N_2$ is the regular neighborhood of $P_2$,
\( N_2 \neq I^n, \quad \pi_1(Bd N_2) = \pi_1(N_2 - P_2) \neq 1 \) and \( N_2 \times I \) is a combinatorial \((n+1)\)-ball. Moreover \( P \times I \cap 0 \).

**Proof of Theorem 5.** The proof will be by induction. For \( n = 4 \) the result follows from Theorem 2, Lemma 5, and Theorem 3. Suppose inductively for \( n = k \) we have a contractible \((k-2)\)-complex \( \rho^{k-2} \), two embeddings \( P_2^{k-2} \) and \( P_1^{k-2} \) in \( S^k \) such that \( N_2^k \approx I^k \) and \( \pi_1(S^k - P_2^{k-2}) = 1 \), while \( \pi_1(S^k - P_1^{k-2}) \neq 1, \quad N_2^k \neq I^k \), \( \pi_1(Bd N_2^k) = \pi_1(N_2^k - P_1^{k-2}) \neq 1 \) and \( N_2^k \times I \approx I^{k+1} \). Also assume \( P_1^{k-2} \times I \cap 0 \).

Using Lemma 3 we get a contractible \((k-1)\)-complex \( P_2^{k-2} \approx \Sigma P_1^{k-2} \) in \( S^{k+1} \) such that \( N(P_2^{k-1}, S^{k+1}) = I^{k+1} \) and \( \pi_1(S^{k+1} - P_2^{k-1}) = 1 \). Using Lemma 4 we get a contractible \((k-1)\)-complex \( P_2^{k-1} \approx \Sigma P_2^{k-2} \) in \( S^{k+1} \) such that \( \pi_1(S^{k+1} - P_2^{k-2}) \neq 1 \). Lemma 5 then implies that \( N_2^{k+1} \neq I^{k+1}, \quad \pi_1(N_2^{k+1} - P_2^{k-2}) = \pi_1(Bd N_2^{k+1}) \neq 1 \). Since \( P_2^{k+1} \approx \Sigma P_2^{k-2} \) and \( P_2^{k-2} \times I \cap 0 \), the third remark in the proof of Lemma 6 gives us that \( P_2^{k-2} \times X \cap 0 \). Also, Lemma 6 gives us that \( N(P_2^{k-1}, S^{k+1}) \times I \approx I^{k+2} \). Finally, since \( P_2^{k-2} \approx P_2^{k-2} \) and \( P_2^{k-1} \approx \Sigma P_2^{k-2} \) \((i = 1, 2)\) we have that \( P_2^{k-2} \neq I^{k-2} \).

**Corollary 6.** For \( n \geq 4 \) there exists a contractible \((n-1)\)-complex \( K^{n-1} \) in \( S^n \) such that \( N(K, S^n) \neq I^n, \quad \pi_1(Bd N(K, S^n)) \neq 1 \) and \( N(K, S^n) \times I \approx I^{n+1} \). Also \( \pi_1(S^n - K^{n-1}) \neq 1 \).

**Corollary 7.** For \( n \geq 4 \) there exists a contractible \( n \)-complex \((combinatorial \quad n\)-manifold with boundary) \( N^n \) in \( S^n \) such that \( N^n \neq I^n, \quad \pi_1(Bd N^n) \neq 1 \) and \( N^n \times I \approx I^{n+1} \). Also \( \pi_1(S^n - N^n) \neq 1 \).

Corollary 7 follows from Theorem 5 by taking \( N^n = N_2 \) of that theorem; Corollary 6 by reducing \( N^n \) to \( K^{n-1} \) using Whitehead elementary contractions and the fact that \( N(K^{n-1}, S^n) \approx N^n, \quad \pi_1(S^n - N^n) \neq 1 \) since \( \pi_1(Cl(S^n - N^n)) \neq 1 \) by Lemma 5. \( \pi_1(S^n - K^{n-1}) \neq 1 \) since we can assume that \( K^{n-1} \subset int N^n \) and hence \( S^n - K^{n-1} \) is of the same homotopy type as \( Cl(S^n - N^n) \) (using Lemma 1).

**Theorem 6.** For \( n \geq 5, \quad N^n_2 \) \((of \quad Theorem \quad 5)\) is a contractible combinatorial \( n \)-manifold with boundary which is not topological \( I^n \), but is combinatorially equivalent to the union of two combinatorial \( n \)-balls, \( B^n_1 \cup B^n_2 \) such that \( B^n_1 \cap B^n_2 \approx B^n_2 \) a combinatorial \( n \)-ball which is a subcomplex of each. Furthermore, \( \int N^n_2 \approx X \cup Y \) where \( X \approx Y \approx X \cap Y \approx E^n \), while \( \int N^n_2 \neq E^n \).

**Proof of Theorem 6.** For \( n \geq 5, \quad N^n_2 \approx N(P_2^{n-2}, S^n) \approx N(\Sigma P_2^{n-2}, S^n) \). Also, \( N^n_2 = N(P_2^{n-2}, S^{n-1}) \neq I^{n-1} \) with \( N^n_2 \times I \approx I^n \). We observe that \( N^n_2 \approx N(C + P_2^{n-2}, S^n) \cup N(C - P_2^{n-2}, S^n) \). Since \( C + P_2^{n-2} \neq 0 \) and \( C - P_2^{n-2} \neq 0 \), Theorem 23\(\) [12] gives us that \( N(C + P_2^{n-2}, S^n) \approx B^n_1 \) and
\( N(C-P^{n-3}_2, S^n) \approx B^n_2 \), the two desired combinatorial \( n \)-balls. Since \( S^n \) was obtained as \( \Sigma B^{n-1} \cup C(\text{Bd } \Sigma B^{n-1}) \) where \( N^{n-1}_2 = N(P^{n-3}_2, S^{n-1}) \) lies in int \( B^{n-1} \), for some \( B^{n-1} \subset S^{n-1} \), it follows that \( B^n_1 \cap B^n_2 \approx N(C+P^{n-3}_2, S^n) \cap N(C-P^{n-3}_2, S^n) \approx N(P^{n-3}_2, S^n) \approx N(P^{n-3}_2, S^{n-1}) \times I \)

which is \( \approx I^n \), that is, our \( B^n_3. N(P^{n-3}_2, S^n) \approx N(P^{n-3}_2, S^{n-1}) \times I \)

since the latter expression is clearly a regular neighborhood of \( P^{n-3}_2 \) in \( S^n \) and any two regular neighborhoods of the same complex are combinatorially equivalent.

Letting \( X = \text{int } B^n_1, Y = \text{int } B^n_2 \), then \( X \cap Y \approx \text{int } B^n_2 \) so that int \( N^n_2 \approx X \cup Y \) and \( X \approx Y \approx X \cap Y \approx E^n \). We have that int \( N^n_2 \not\approx E^n \) since \( N^n_2 \) is an \( n \)-manifold with boundary (and hence collared from the inside [3]) and \( \pi_1(\text{Bd } N^n_2) \not= 1. \) That is, if int \( N^n_2 = E^n \), simple closed curves near “infinity” can be shrunk near “infinity,” but \( \text{Bd } N^n_2 \times [0, 1) \) the collar of \( \text{Bd } N^n_2 \) in \( N^n_2 \) is not simply connected and hence there exist nontrivial simple closed curves in \( \text{Bd } N^n_2 \times (0, 1) \).

**Theorem 7.** Suppose \( C' \) is a contractible \( k \)-complex that can be \( \text{p.w.l.} \) embedded in a combinatorial \( n \)-sphere \( S^n \) with triangulation \( T \) as a subcomplex \( C \) such that \( \pi_1(S^n-C) \not= 1 \) (necessarily \( k=n, n-1, \) or \( n-2 \) by Lemma 5 and Theorem 1 [5]). Then for \( n \geq 5 \) there exists a \( \text{p.w.l.} \) embedding \( \hat{C} \) of \( C' \) in \( S^n \) under \( T \) such that \( N(\hat{C}, S^n) \approx N(C, S^n)(\not\approx I^n) \), but now \( \pi_1(S^n-\hat{C}) = 1. \)

**Proof of Theorem 7.** Let \( \Sigma \) be the combinatorial \( n \)-manifold formed by attaching two copies of \( N(C, S^n) \) together along their boundaries. Since \( N(C, S^n) \) is contractible, \( \Sigma \) is a combinatorial \( n \)-manifold with the homotopy type of \( S^n \). Hence for \( n \geq 5, \Sigma \) is a topological \( n \)-sphere which is also a combinatorial \( n \)-manifold [10], [14]. Let us also denote \( C \) in \( \Sigma \) as the complex \( C \) in one copy of \( N(C, S^n) \) used in forming \( \Sigma \). Now \( \pi_1(\Sigma - C) = 1 \) since \( \Sigma - C = \{ [0, 1) \times \text{Bd } N(C, S^n) \} \cup N(C, S^n) \) which is homotopically equivalent to \( N(C, S^n) \). Let \( p \) be an interior point of some \( n \)-simplex of \( \Sigma \) missing the copy of \( N(C, S^n) \) in \( \Sigma \) containing \( C \). Now \( \Sigma - \{ p \} \) is p.w.l. equivalent to \( S^n - \{ q \} \) under \( T \) for some \( q \in S^n \) since \( n \geq 5 \) [11]. Hence there exists a p.w.l. homeomorphism \( h \) of \( \Sigma - \{ p \} \) onto \( S^n - \{ q \} \) taking \( C \) and \( N(C, S^n) \) (as in \( \Sigma - \{ p \} \)) into \( S^n - \{ q \} \) (under \( T \)). Then \( h(C) = \hat{C} \) is a p.w.l. embedding of \( C' \) in \( S^n \) and \( \pi_1(S^n-\hat{C}) = 1 \) since \( \pi_1(\Sigma - C) = 1. \) Since \( h(N(C, S^n)) \) is a regular neighborhood of \( \hat{C} \) in \( S^n \) under a subdivision of \( T, N(\hat{C}, S^n) \approx h(N(C, S^n)) \approx N(C, S^n). \) Note, if \( C' \) is the contractible \( k \)-complex given in Theorem 5, then one has that \( \Sigma \) is in fact a combinatorial \( n \)-sphere (since \( N \times I \approx I^{n+1} \)). Hence \( \Sigma \approx S^n \) under \( T \) and the result follows immediately.
Bibliography

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