AN \textit{L}1 \textsc{extremal} problem \textsc{for} \textsc{polynomials}

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There are a large number of \textsc{polynomial} extremal \textsc{problems} \textsc{all} of which \textsc{seem} to \textsc{ask} \textsc{the} \textsc{same} \textsc{question}. In very crude terms it is the following one: \textsc{How} \textsc{closely} \textsc{can} \textsc{we} \textsc{approximate} \textsc{a} \textsc{nontrivial} \textsc{situation} where

\[ |a_k| \text{ is a constant for } k = 0, 1, 2, \ldots, n \]

and

\[ \left| \sum_{k=0}^{n} a_k z^k \right| \text{ is a constant for } |z| = 1 ? \]

One such extremal \textsc{problem} \textsc{which} has \textsc{received} \textsc{considerable} \textsc{attention} is the following one:

\textsc{To choose} \textsc{a} \textsc{k} \textsc{with} \textsc{|a_k|} \textsc{= 1} \textsc{so that} \textsc{M} = \textsc{max}_{|z|=1} \left| \sum_{k=0}^{n} a_k z^k \right| \textsc{is a minimum.}

\textsc{Since}

\[ M^2 \geq \frac{1}{2\pi} \int \left| \sum_{k=0}^{n} a_k z^k \right|^2 |dz| = \sum_{k=0}^{n} |a_k|^2 = n + 1 \]

\textsc{it follows that} \textsc{M} \textsc{\geq (n+1)^{1/2}}. \textsc{However}, \textsc{in} \textsc{order} \textsc{that} \textsc{(n+1)^{1/2}} \textsc{be} \textsc{(close} \textsc{to)} \textsc{the} \textsc{right} \textsc{answer} \textsc{the} \textsc{inequality} \textsc{M}^2 \geq (1/(2\pi))\int |\sum a_k z^k|^2 |dz| \textsc{must be} \textsc{(close} \textsc{to)} \textsc{equality}, \textsc{which} \textsc{is} \textsc{to} \textsc{say,} \textsc{|} \sum a_k z^k \textsc{|} \textsc{must be close} \textsc{to} \textsc{constant.}

\textsc{It was proved} \textsc{by Hardy} \textsc{that} \textsc{this} \textsc{minimum} \textsc{M}, \textsc{M}_n, \textsc{satisfies} \textsc{M}_n \leq c(n+1)^{1/2}, \textsc{c some} \textsc{absolute} \textsc{constant (see Zygmund [4])}. \textsc{Shapiro} \textsc{has} \textsc{even} \textsc{shown} \textsc{that} \textsc{the} \textsc{a} \textsc{k} \textsc{can} \textsc{be} \textsc{chosen} \textsc{real (i.e.,} \textsc{equal} \textsc{to} \textsc{±1)} \textsc{and} \textsc{the} \textsc{same} \textsc{estimate} \textsc{achieved (see Rudin [3]).}

\textsc{Thus} \textsc{the} \textsc{order} \textsc{of} \textsc{magnitude} \textsc{of} \textsc{M}_n \textsc{is} \textsc{determined} \textsc{as} \textsc{√n}. \textsc{The} \textsc{deeper} \textsc{question} \textsc{regarding} \textsc{the} \textsc{limit} \textsc{of} \textsc{M}_n/\sqrt{n} \textsc{remains} \textsc{unsettled.} \textsc{In} \textsc{terms} \textsc{of} \textsc{our} \textsc{original} \textsc{heuristic} \textsc{formulation} \textsc{it makes} \textsc{a} \textsc{vital} \textsc{difference} \textsc{whether} \textsc{M}_n/\sqrt{n} \textsc{→} 1 \textsc{or} \textsc{not}. \textsc{Some} \textsc{partial} \textsc{results} \textsc{in} \textsc{this} \textsc{direction} \textsc{have} \textsc{been} \textsc{obtained} \textsc{by Erdős and Littlewood [1].}

\textsc{Another} \textsc{extremal} \textsc{problem} \textsc{in} \textsc{the} \textsc{same} \textsc{spirit} \textsc{is} \textsc{the} \textsc{following:}

\textsc{Among} \textsc{all} \textsc{polynomials} \textsc{for} \textsc{which}

\[ \left| \sum_{k=0}^{n} a_k z^k \right| \leq 1 \text{ for } |z| \leq 1. \]

\textsc{To} \textsc{find} \textsc{max}_{a_k} \sum_{k=0}^{n} |a_k| = M_n.

\textsc{Here} \textsc{the} \textsc{Schwarz} \textsc{inequality} \textsc{gives}

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so that $\mathcal{M}_n \leq (n+1)^{1/2}$. Again for this estimate to be accurate we must have near equality in all the above estimates. Thus $|a_k|$ should be nearly constant, as should $|\sum a_k z^k|$.

It was shown by Newman that, for $n > 0$, $\mathcal{M}_n \leq \sqrt{n}$, equality holding for $n = 1, 2, 4$. Shapiro, however, showed that $n = 1, 2, 4$ are the only cases of equality. Hardy’s example shows that $\mathcal{M}_n \geq c \sqrt{n}$, $c > 0$, and so settles the order of magnitude of $\mathcal{M}_n$. Again the deeper problem of whether $\mathcal{M}_n/\sqrt{n} \rightarrow 1$ remains unsettled.

As our third example we consider the problem of maximizing

$$\sum_{k=0}^{n} |a_k| \leq \left( (n+1) \sum_{k=0}^{n} |a_k|^2 \right)^{1/2} = \left( \frac{n+1}{2\pi} \int_{|z|=1} \left| \sum_{k=0}^{n} a_k z^k \right|^2 \, dz \right)^{1/2} \leq (n+1)^{1/2}$$

subject to $|a_k| = 1$ for $k = 0, 1, \ldots, n$. If this maximum is called $I_n$, we obtain $I_n \leq (n+1)^{1/2}$ by the Schwarz inequality. In the case of real $a_t$ this has been improved to $I_n \leq (n+0.97)^{1/2}$ (for $n > 0$) (see Newman [2]). Once more, by the previously cited examples, it follows that $I_n > c \sqrt{n}$, $c > 0$, and so the order of magnitude of $I_n$ is again $\sqrt{n}$.

The purpose of the present paper is to prove that $I_n/\sqrt{n} \rightarrow 1$, so that, at least in this sense, the meta-problem posed in our introduction is solved. We actually prove somewhat more, namely

**Theorem 1.** $I_n \geq \sqrt{n} - c$, $c$ an absolute constant.

The example we have constructed is related to the Gaussian sums and is motivated by the fact that these sums have size $\sqrt{n}$ while consisting of $n$ terms of modulus 1. We set $\omega = \exp(\pi i/(n+1))$, $a_k = \omega^k$, $k = 0, 1, \ldots, m$, $P(z) = \sum_{k=0}^{n} a_k z^k$, and we prove

**Lemma.** $(1/2\pi) \int_{|z|=1} |P(z)|^4 \, dz = n^2 + O(n^{1/2})$.

**Proof.** Writing $|\sum_{k=0}^{n} a_k e^{i\theta_k}|^2 = \sum_{k=0}^{n} c_j e^{i\theta_j}$, we note that $(1/2\pi) \int_{|z|=1} |P(z)|^4 \, dz = \sum_{k=0}^{n} |c_j|^2$. We also see that $c_0 = n+1$. Next we examine $c_j$ for $j > 0$. We have

$$c_j = \sum_{k=0}^{n-j} \omega^{-j+k-j} = \omega^j \sum_{k=0}^{n-j} \omega^{k-j} = \omega^j \frac{1 - \omega^{n-j+j}}{1 - \omega^j} = -\omega^{-j} \frac{\omega^j - \omega^{-j}}{\omega^j - \omega^{-j}}$$

and we obtain the formula
\[
|c_j|^2 = \frac{\sin^2 j \pi}{n + 1} \quad \text{for } j = 1, 2, \ldots, n.
\]

We proceed to estimate \( \sum_{j=1}^{n} |c_j|^2 \) by splitting this sum into four parts

\[
S_1 = \sum_{j \in \mathbb{N}} |c_j|^2, \quad S_2 = \sum_{\sqrt{n} < j < (n+1)/2} |c_j|^2, \\
S_3 = \sum_{(n+1)/2 < j < n+1-\sqrt{n}} |c_j|^2, \quad S_4 = \sum_{n+1-\sqrt{n} \leq j \leq n} |c_j|^2.
\]

First of all we note that, for \( j > 0 \) by (1)

\[
|c_{n+1-j}|^2 = |c_j|^2
\]

so that

\[
|S_3| \leq |S_2| \quad \text{and} \quad |S_4| \leq |S_1|.
\]

Next, from the inequality \( |\sin j\theta/\sin \theta| \leq j \), we conclude that

\[
|S_1| \leq \sum_{j \in \mathbb{N}} j^2 \leq n \sum_{j \in \mathbb{N}} 1 \leq n^{3/2}.
\]

Furthermore, from the inequality \( |1/\sin \theta| \leq \pi/2\theta \) (valid in \( 0 \leq \theta \leq \pi/2 \)) we conclude that

\[
|S_2| \leq \sum_{\sqrt{n} < j < (n+1)/2} \frac{(n+1)^2}{4j^2} < \frac{(n+1)^2}{4} \sum_{\sqrt{n} < j < \infty} \frac{1}{j^2} \leq \frac{n^2}{\sqrt{n}} = n^{3/2}.
\]

If we observe that \( \tilde{c}_{-j} = \tilde{c}_j \), and combine (2), (3), and (4), we finally obtain

\[
\sum_{j \neq 0} |c_j|^2 \leq 8n^{3/2}
\]

and the lemma is proved.

Proof of Theorem 1. We use the Hölder inequality to conclude that

\[
\int |f|^2 \leq \left( \int |f|^4 \right)^{1/4} \left( \int |f|^2 \right)^{3/4}
\]

which, applied to our present case, gives
\[ \frac{1}{2\pi} \int_{|z|=1} |P(z)| \geq \left( \frac{1}{2\pi} \int |P(z)|^2 \right)^{1/2} = \frac{(n+1)^{3/2}}{\left( \frac{1}{2\pi} \int |P(z)|^4 \right)^{1/4}} \]

so that, applying our lemma, we obtain

\[ \frac{1}{2\pi} \int |\sum a_k z^k| \geq \frac{(n+1)^{3/2}}{(n^2 + An^{3/2})^{1/2}} > \frac{n^{1/2}}{(1 + An^{-1/2})^{1/2}} > n^{1/2}(1 - cn^{-1/2}) = \sqrt{n} - c. \]

Q.E.D.

Theorem 1 is a statement regarding \( L^1 \) norms. By applying the principle of duality we can obtain from it a theorem on \( L^\infty \) functions. Because of its independent interest we record the statement of this theorem.

**Theorem 2.** Given \( n \geq 0 \), there exists a measurable function, \( f(\theta) \), with period \( 2\pi \) and satisfying \( |f(\theta)| \leq 1 \) for all \( \theta \), such that, with \( \{b_k\} \) the Fourier coefficients of \( f(\theta) \), we have \( \sum_{k=0}^{n} |b_k| \geq \sqrt{n} - c \), \( c \) an absolute constant.

One further remark is perhaps pertinent. We point out, namely, that the behavior exhibited by our polynomial, \( P(z) \), is quite exceptional. Indeed, the crucial fact about \( P(z) \) is that its \( L^4 \) norm is close to \( \sqrt{n} \), and it follows from the work of Paley (see Zygmund [4]) that, in a certain sense, most \( n \)th degree polynomials with coefficients of modulus 1 have \( L^4 \) norms which, instead, are close to \( 2^{1/4} \sqrt{n} \).

**References**

1. P. Erdős and J. E. Littlewood, Personal oral communication.

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