AN L¹ EXTREMAL PROBLEM FOR POLYNOMIALS

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There are a large number of polynomial extremal problems all of which seem to ask the same question. In very crude terms it is the following one: How closely can we approximate a nontrivial situation where

\[ |a_k| \text{ is a constant for } k = 0, 1, 2, \ldots, n \]

and

\[ \left| \sum_{k=0}^{n} a_k z^k \right| \text{ is a constant for } |z| = 1? \]

One such extremal problem which has received considerable attention is the following one:

To choose \( a_k \) with \( |a_k| = 1 \) so that \( M = \max_{|z|=1} | \sum_{k=0}^{n} a_k z^k | \) is a minimum.

Since

\[ \frac{1}{2\pi} \int | \sum_{k=0}^{n} a_k z^k |^2 |dz| = \sum_{k=0}^{n} |a_k|^2 = n + 1 \]

it follows that \( M \geq (n+1)^{1/2} \). However, in order that \( (n+1)^{1/2} \) be (close to) the right answer the inequality \( M^2 \geq \frac{1}{(2\pi)^2} \int | \sum a_k z^k |^2 |dz| \) must be (close to) equality, which is to say, \( | \sum a_k z^k | \) must be close to constant.

It was proved by Hardy that this minimum \( M, M_n \), satisfies

\[ M_n \leq c(n+1)^{1/2}, \] for some absolute constant (see Zygmund [4]). Shapiro has even shown that the \( a_k \) can be chosen real (i.e., equal to \( \pm 1 \)) and the same estimate achieved (see Rudin [3]).

Thus the order of magnitude of \( M_n \) is determined as \( \sqrt{n} \). The deeper question regarding the limit of \( M_n/\sqrt{n} \) remains unsettled. In terms of our original heuristic formulation it makes a vital difference whether \( M_n/\sqrt{n} \to 1 \) or not. Some partial results in this direction have been obtained by Erdös and Littlewood [1].

Another extremal problem in the same spirit is the following:

Among all polynomials for which

\[ \left| \sum_{k=0}^{n} a_k z^k \right| \leq 1 \text{ for } |z| \leq 1. \]

To find \( \max_{a_k} \sum_{k=0}^{n} |a_k| = M_n \).

Here the Schwarz inequality gives

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\[ \sum |a_k| \leq ((n + 1) \sum |a_k|^2)^{1/2} = \left( \frac{n + 1}{2\pi} \int_{|z|=1} \left| \sum a_k z^k \right|^2 |dz| \right)^{1/2} \leq (n + 1)^{1/2} \]

so that \( M_n \leq (n + 1)^{1/2} \). Again for this estimate to be accurate we must have near equality in all the above estimates. Thus \(|a_k|\) should be nearly constant, as should \(|\sum a_k z^k|\).

It was shown by Newman that, for \( n > 0 \), \( M_n \leq \sqrt{n} \), equality holding for \( n = 1, 2, \) and \( 4 \). Shapiro, however, showed that \( n = 1, 2, 4 \) are the only cases of equality. Hardy’s example shows that \( M_n \geq c \sqrt{n} \), \( c > 0 \), and so settles the order of magnitude of \( M_n \). Again the deeper problem of whether \( M_n / \sqrt{n} \to 1 \) remains unsettled.

As our third example we consider the problem of maximizing

\[ \frac{1}{2\pi} \int \left| \sum_{k=0}^n a_k z^k \right|^2 |dz| \text{ subject to } |a_k| = 1 \text{ for } k = 0, 1, \ldots, n. \]

If this maximum is called \( I_n \), we obtain \( I_n \leq (n + 1)^{1/2} \) by the Schwarz inequality. In the case of real \( a_i \) this has been improved to \( I_n \leq (n + .97)^{1/2} \) (for \( n > 0 \)) (see Newman [2]). Once more, by the previously cited examples, it follows that \( I_n > c \sqrt{n} \), \( c > 0 \), and so the order of magnitude of \( I_n \) is again \( \sqrt{n} \).

The purpose of the present paper is to prove that \( I_n / \sqrt{n} \to 1 \), so that, at least in this sense, the meta-problem posed in our introduction is solved. We actually prove somewhat more, namely

**Theorem 1.** \( I_n \geq \sqrt{n - c} \), \( c \) an absolute constant.

The example we have constructed is related to the Gaussian sums and is motivated by the fact that these sums have size \( \sqrt{n} \) while consisting of \( n \) terms of modulus 1. We set \( \omega = \exp (\pi i / (n + 1)) \), \( a_k = \omega^k \), \( k = 0, 1, \ldots, m \), \( P(z) = \sum_{k=0}^n a_k z^k \), and we prove

**Lemma.** \((1/2\pi) \int_{|z|=1} |P(z)|^4 |dz| = n^2 + O(n^{3/2}).\)

**Proof.** Writing \( | \sum_{k=0}^n a_k e^{i\theta k} |^2 = \sum_{k=0}^n c_j e^{i\theta j} \), we note that \((1/2\pi) \int_{|z|=1} |P(z)|^4 |dz| = \sum_{j=0}^n |c_j|^2 \). We also see that \( c_0 = n + 1 \). Next we examine \( c_j \) for \( j > 0 \). We have

\[ c_j = \sum_{k=0}^{n-j} \omega^{(k+j-k^2)} = \omega^j \sum_{k=0}^{n-j} \omega^{2kj} = \omega^j \left( 1 - \frac{\omega^{2(n+1-j)} j}{1 - \omega^{2j}} \right) = - \omega^{-j} \frac{\omega^j - \omega^{-j}}{\omega^j - \omega^{-j}} \]

and we obtain the formula
\[
(1) \quad |c_j|^2 = \frac{\sin^2 j \frac{\pi}{n+1}}{n+1} \quad \text{for } j = 1, 2, \ldots, n.
\]

We proceed to estimate \( \sum_{j=1}^{n} |c_j|^2 \) by splitting this sum into four parts
\[
S_1 = \sum_{j \leq \sqrt{n}} |c_j|^2, \quad S_2 = \sum_{\sqrt{n} \leq j < (n+1)/2} |c_j|^2,
\]
\[
S_3 = \sum_{(n+1)/2 \leq j < n+1-\sqrt{n}} |c_j|^2, \quad S_4 = \sum_{n+1-\sqrt{n} \leq j \leq n} |c_j|^2.
\]

First of all we note that, for \( j > 0 \) by (1)
\[
|c_{n+1-j}|^2 = |c_j|^2
\]
so that
\[
(2) \quad |S_3| \leq |S_2| \quad \text{and} \quad |S_4| \leq |S_1|.
\]

Next, from the inequality \( |\sin j\theta/\sin \theta| \leq j \), we conclude that
\[
(3) \quad |S_1| \leq \sum_{j \leq \sqrt{n}} j^2 \leq n \sum_{j \leq \sqrt{n}} 1 \leq n^{3/2}.
\]

Furthermore, from the inequality \( |1/\sin \theta| \leq \pi/2\theta \) (valid in \( 0 \leq \theta \leq \pi/2 \)) we conclude that
\[
(4) \quad |S_2| \leq \sum_{\sqrt{n} \leq j < (n+1)/2} \frac{(n+1)^2}{4j^2} < \frac{(n+1)^2}{4} \sum_{\sqrt{n} < j < \infty} \frac{1}{j^2} \leq \frac{n^2}{\sqrt{n}} = n^{3/2}.
\]

If we observe that \( c_{-j} = \bar{c}_j \), and combine (2), (3), and (4), we finally obtain
\[
(5) \quad \sum_{j \neq 0} |c_j|^2 \leq 8n^{3/2}
\]
and the lemma is proved.

**Proof of Theorem 1.** We use the Hölder inequality to conclude that
\[
\int |f|^2 \leq \left( \int |f|^4 \right)^{1/2} \left( \int |f|^2 \right)^{1/2}
\]
which, applied to our present case, gives
\[
\frac{1}{2\pi} \int_{|z|=1} |P| \geq \left( \frac{\frac{1}{2\pi} \int |P|^2}{\left( \frac{1}{2\pi} \int |P|^4 \right)^{1/2}} \right)^{1/2} = \frac{(n + 1)^{3/2}}{(1 + An^{-1/2})^{1/2}} > n^{1/2}(1 - cn^{-1/2}) = \sqrt{n} - c.
\]
so that, applying our lemma, we obtain

\[
\frac{1}{2\pi} \int |\sum b_k z^k| \geq \frac{(n + 1)^{3/2}}{(n^2 + An^{3/2})^{1/2}} > \frac{n^{1/2}}{(1 + An^{-1/2})^{1/2}} > n^{1/2}(1 - cn^{-1/2}) = \sqrt{n} - c.
\]

Q.E.D.

Theorem 1 is a statement regarding \(L^1\) norms. By applying the principle of duality we can obtain from it a theorem on \(L^\infty\) functions. Because of its independent interest we record the statement of this theorem.

**Theorem 2.** Given \(n \geq 0\), there exists a measurable function, \(f(\theta)\), with period \(2\pi\) and satisfying \(|f(\theta)| \leq 1\) for all \(\theta\), such that, with \(\{b_k\}\) the Fourier coefficients of \(f(\theta)\), we have \(\sum_{k=0}^n |b_k| \leq \sqrt{n} - c\), \(c\) an absolute constant.

One further remark is perhaps pertinent. We point out, namely, that the behavior exhibited by our polynomial, \(P(z)\), is quite exceptional. Indeed, the crucial fact about \(P(z)\) is that its \(L^4\) norm is close to \(\sqrt{n}\), and it follows from the work of Paley (see Zygmund [4]) that, in a certain sense, most \(n\)th degree polynomials with coefficients of modulus 1 have \(L^4\) norms which, instead, are close to \(2^{1/4} \sqrt{n}\).

**References**

1. P. Erdös and J. E. Littlewood, Personal oral communication.

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