

AN L^1 EXTREMAL PROBLEM FOR POLYNOMIALS

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There are a large number of polynomial extremal problems all of which seem to ask the same question. In very crude terms it is the following one: How closely can we approximate a nontrivial situation where

$$|a_k| \text{ is a constant for } k = 0, 1, 2, \dots, n$$

and

$$\left| \sum_{k=0}^n a_k z^k \right| \text{ is a constant for } |z| = 1?$$

One such extremal problem which has received considerable attention is the following one:

To choose a_k with $|a_k| = 1$ so that $M = \max_{|z|=1} \left| \sum_{k=0}^n a_k z^k \right|$ is a minimum.

Since

$$M^2 \geq \frac{1}{2\pi} \int \left| \sum_{k=0}^n a_k z^k \right|^2 |dz| = \sum_{k=0}^n |a_k|^2 = n + 1$$

it follows that $M \geq (n+1)^{1/2}$. However, in order that $(n+1)^{1/2}$ be (close to) the right answer the inequality $M^2 \geq (1/2\pi) \int \left| \sum_{k=0}^n a_k z^k \right|^2 |dz|$ must be (close to) equality, which is to say, $\left| \sum_{k=0}^n a_k z^k \right|$ must be close to constant.

It was proved by Hardy that this minimum M , M_n , satisfies $M_n \leq c(n+1)^{1/2}$, c some absolute constant (see Zygmund [4]). Shapiro has even shown that the a_k can be chosen real (i.e., equal to ± 1) and the same estimate achieved (see Rudin [3]).

Thus the order of magnitude of M_n is determined as \sqrt{n} . The deeper question regarding the limit of M_n/\sqrt{n} remains unsettled. In terms of our original heuristic formulation it makes a vital difference whether $M_n/\sqrt{n} \rightarrow 1$ or not. Some partial results in this direction have been obtained by Erdős and Littlewood [1].

Another extremal problem in the same spirit is the following:

Among all polynomials for which

$$\left| \sum_{k=0}^n a_k z^k \right| \leq 1 \text{ for } |z| \leq 1.$$

To find $\max_{\{a_k\}} \sum_{k=0}^n |a_k| = \mathfrak{N}_n$.

Here the Schwarz inequality gives

Received by the editors November 3, 1964.

$$\sum |a_k| \leq ((n + 1) \sum |a_k|^2)^{1/2} = \left(\frac{n + 1}{2\pi} \int_{|z|=1} \left| \sum a_k z^k \right|^2 |dz| \right)^{1/2} \leq (n + 1)^{1/2}$$

so that $\mathfrak{M}_n \leq (n + 1)^{1/2}$. Again for this estimate to be accurate we must have near equality in all the above estimates. Thus $|a_k|$ should be nearly constant, as should $|\sum a_k z^k|$.

It was shown by Newman that, for $n > 0$, $\mathfrak{M}_n \leq \sqrt{n}$, equality holding for $n = 1, 2$, and 4 . Shapiro, however, showed that $n = 1, 2, 4$ are the only cases of equality. Hardy's example shows that $\mathfrak{M}_n \geq c\sqrt{n}$, $c > 0$, and so settles the order of magnitude of \mathfrak{M}_n . Again the deeper problem of whether $\mathfrak{M}_n/\sqrt{n} \rightarrow 1$ remains unsettled.

As our third example we consider the problem of maximizing

$$\frac{1}{2\pi} \int \left| \sum_{k=0}^n a_k z^k \right| |dz| \text{ subject to } |a_k| = 1 \text{ for } k = 0, 1, \dots, n.$$

If this maximum is called I_n , we obtain $I_n \leq (n + 1)^{1/2}$ by the Schwarz inequality. In the case of real a_i this has been improved to $I_n \leq (n + .97)^{1/2}$ (for $n > 0$) (see Newman [2]). Once more, by the previously cited examples, it follows that $I_n > c\sqrt{n}$, $c > 0$, and so the order of magnitude of I_n is again \sqrt{n} .

The purpose of the present paper is to prove that $I_n/\sqrt{n} \rightarrow 1$, so that, at least in this sense, the meta-problem posed in our introduction is solved. We actually prove somewhat more, namely

THEOREM 1. $I_n \geq \sqrt{n} - c$, c an absolute constant.

The example we have constructed is related to the Gaussian sums and is motivated by the fact that these sums have size \sqrt{n} while consisting of n terms of modulus 1. We set $\omega = \exp(\pi i/(n + 1))$, $a_k = \omega^{k^2}$, $k = 0, 1, \dots, m$, $P(z) = \sum_{k=0}^n a_k z^k$, and we prove

LEMMA. $(1/2\pi) \int_{|z|=1} |P(z)|^4 |dz| = n^2 + O(n^{3/2})$.

PROOF. Writing $|\sum_{k=0}^n a_k e^{ik\theta}|^2 = \sum_{-n}^n c_j e^{ij\theta}$, we note that $(1/2\pi) \int_{|z|=1} |P(z)|^4 |dz| = \sum_{-n}^n |c_j|^2$. We also see that $c_0 = n + 1$. Next we examine c_j for $j > 0$. We have

$$c_j = \sum_{k=0}^{n-j} \omega^{(k+j)^2 - k^2} = \omega^{j^2} \sum_{k=0}^{n-j} \omega^{2kj} = \omega^{j^2} \frac{1 - \omega^{2(n+1-j)j}}{1 - \omega^{2j}} = -\omega^{-j} \frac{\omega^{j^2} - \omega^{-j^2}}{\omega^j - \omega^{-j}}$$

and we obtain the formula

$$(1) \quad |c_j|^2 = \frac{\sin^2 j^2 \frac{\pi}{n+1}}{\sin^2 j \frac{\pi}{n+1}} \quad \text{for } j = 1, 2, \dots, n.$$

We proceed to estimate $\sum_{j=1}^n |c_j|^2$ by splitting this sum into four parts

$$\begin{aligned} S_1 &= \sum_{j \leq \sqrt{n}} |c_j|^2, & S_2 &= \sum_{\sqrt{n} < j \leq (n+1)/2} |c_j|^2, \\ S_3 &= \sum_{(n+1)/2 < j < n+1-\sqrt{n}} |c_j|^2, & S_4 &= \sum_{n+1-\sqrt{n} \leq j \leq n} |c_j|^2. \end{aligned}$$

First of all we note that, for $j > 0$ by (1)

$$|c_{n+1-j}|^2 = |c_j|^2$$

so that

$$(2) \quad |S_3| \leq |S_2| \quad \text{and} \quad |S_4| \leq |S_1|.$$

Next, from the inequality $|\sin j\theta/\sin \theta| \leq j$, we conclude that

$$(3) \quad |S_1| \leq \sum_{j \leq \sqrt{n}} j^2 \leq n \sum_{j \leq \sqrt{n}} 1 \leq n^{3/2}.$$

Furthermore, from the inequality $|1/\sin \theta| \leq \pi/2\theta$ (valid in $0 \leq \theta \leq \pi/2$) we conclude that

$$(4) \quad |S_2| \leq \sum_{\sqrt{n} < j \leq (n+1)/2} \frac{(n+1)^2}{4j^2} < \frac{(n+1)^2}{4} \sum_{\sqrt{n} < j < \infty} \frac{1}{j^2} \leq \frac{n^2}{\sqrt{n}} = n^{3/2}.$$

If we observe that $\dot{c}_{-j} = \bar{c}_j$, and combine (2), (3), and (4), we finally obtain

$$(5) \quad \sum_{j \neq 0} |c_j|^2 \leq 8n^{3/2}$$

and the lemma is proved.

PROOF OF THEOREM 1. We use the Hölder inequality to conclude that

$$\int |f|^2 \leq \left(\int |f|^4 \right)^{1/2} \left(\int |f| \right)^{2/3}$$

which, applied to our present case, gives

$$\frac{1}{2\pi} \int_{|z|=1} |P| \geq \frac{\left(\frac{1}{2\pi} \int |P|^2\right)^{3/2}}{\left(\frac{1}{2\pi} \int |P|^4\right)^{1/2}} = \frac{(n+1)^{3/2}}{\left(\frac{1}{2\pi} \int |P|^4\right)^{1/2}}$$

so that, applying our lemma, we obtain

$$\begin{aligned} \frac{1}{2\pi} \int \left| \sum a_k z^k \right| &\geq \frac{(n+1)^{3/2}}{(n^2 + An^{3/2})^{1/2}} > \frac{n^{1/2}}{(1 + An^{-1/2})^{1/2}} \\ &> n^{1/2}(1 - cn^{-1/2}) = \sqrt{n} - c. \end{aligned} \quad \text{Q.E.D.}$$

Theorem 1 is a statement regarding L^1 norms. By applying the principle of duality we can obtain from it a theorem on L^∞ functions. Because of its independent interest we record the statement of this theorem.

THEOREM 2. *Given $n \geq 0$, there exists a measurable function, $f(\theta)$, with period 2π and satisfying $|f(\theta)| \leq 1$ for all θ , such that, with $\{b_k\}$ the Fourier coefficients of $f(\theta)$, we have $\sum_{k=0}^n |b_k| \geq \sqrt{n} - c$, c an absolute constant.*

One further remark is perhaps pertinent. We point out, namely, that the behavior exhibited by our polynomial, $P(z)$, is quite exceptional. Indeed, the crucial fact about $P(z)$ is that its L^4 norm is close to \sqrt{n} , and it follows from the work of Paley (see Zygmund [4]) that, in a certain sense, *most* n th degree polynomials with coefficients of modulus 1 have L^4 norms which, instead, are close to $2^{1/4}\sqrt{n}$.

REFERENCES

1. P. Erdős and J. E. Littlewood, Personal oral communication.
2. D. J. Newman, *Norms of polynomials*, Amer. Math. Monthly **67** (1960), 778–779.
3. W. Rudin, *Some theorems on Fourier coefficients*, Proc. Amer. Math. Soc. **10** (1959), 855–859.
4. A. Zygmund, *Trigonometric series*, Vol. I, Cambridge Univ. Press, New York, 1959; p. 199, Theorem 4.7.

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