SOME REMARKS ON BOUNDARY VALUE PROBLEMS

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Schechter [6] considers the problem of finding necessary and sufficient conditions for the existence of solutions in various spaces to general boundary value problems for an arbitrary partial differential operator $A$. The boundary conditions are used to determine certain subspace $V^{q,p}$ of $H^{q,p}$. The conditions are stated in terms of certain inequalities under conditions which are satisfied by general elliptic boundary value problems for regions $\Omega$ relatively compact in $\mathbb{R}^n$. The main tool is a representation theorem for continuous linear functionals on certain subspaces of $H^{q,p}$. The basic assumption is that the kernel $N'$ of the adjoint operator $A'$ is finite dimensional.

If $\Omega$ is relatively compact the a priori inequalities as proved by Agmon, Douglis and Nirenberg [1]; Schechter [5]; and Browder [2] for example applied to $A'$ together with Rellich's lemma yield the finite dimensionality of $N'$. The a priori estimates are true on regions $\Omega$ which are not necessarily relatively compact but then we do not know $N'$ is finite dimensional. It seems of interest therefore to know that at least in the Hilbert space setting Schechter's results are true even if $N'$ is not finite dimensional. The a priori inequality tells us that on $N', H^{2m,p}$ and $H^{0,p}$ induce the same topology. In what follows we show that when $p = 2$, this is all we need to know to obtain Schechter's representation theorem. These results can be stated abstractly and we do so here.

If $E$ and $F$ are two topological vector spaces we use the notation $E \subseteq F$ to mean that (i) $E$ is a subset of $F$ and (ii) the canonical injection of $E$ into $F$ is continuous, i.e., that $E$ has a finer topology than $F$. We use the term $E$ is dense in $F$ to mean that $E$ with topology induced on it by $F$ is dense in $F$. Finally we use the notation $\mathfrak{L}(E, F)$ for the set of continuous linear maps of $E$ into $F$.

In what follows $H^0$ and $H'$ will be Hilbert spaces with $H' \subseteq H^0$, $H'$ dense in $H^0$. For convenience we will suppose that $\|u\|_0 \leq \|u\|_1$ for $u \in H'$. $N$ will be a closed subspace of $H'$ for which there exists a constant $c_0$ such that $\|u\|_1 \leq c_0 \|u\|_0$ for $u \in N$. Thus $N$ is also a closed subspace of $H^0$ and $H^0$ and $H'$ induce the same topology on $N$.

1. Theorem. There is a positive-definite, self-adjoint operator $B$ with domain equal to $H'$ such that for $u \in H', \|u\|_1 = \|Bu\|_0$.

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This follows from a well-known result of Friedrichs [4] and the Spectral Theorem.

Again using the Spectral Theorem for $0 < t < 1$ we let $H^t$ be the domain of $B^t$ and for $u \in H^t$ we set $\|u\|_t = \|B^tu\|_0$. The $H^t$ are called interpolation spaces.

2. **Proposition.** With the norms $\|\cdot\|_t$, the $H^t$ for $0 < t < 1$ are Hilbert spaces and for $0 < s < t < 1$, $H^t \subset H^s$ with $H^t$ dense in $H^s$.

3. **Theorem (Lions [3]).** Let $K^0$ and $K'$ be another pair of Hilbert spaces with $K' \subset K^0$ and $K'$ dense in $K^0$. Let $K^t$, $0 < t < 1$ be the analogous interpolation spaces. If $T \in \mathcal{L}(H^0, K^0)$ and $T \in \mathcal{L}(H^t, K')$ then for $0 < t < 1$, $T \in \mathcal{L}(H^t, K^t)$.

4. **Definition.** For $u \in H^0$ let

$$\|u\|_{-t} = \sup \{|(u, v)| : v \in H^t \text{ and } \|v\|_t \leq 1\}.$$  

Let $H^{-t}$ be the completion of $H^0$ in the norm $\|\cdot\|_{-t}$.

5. **Theorem.** There is a canonical topological isomorphism between $H^{-t}$ and the dual space of $H^t$ for $0 \leq t \leq 1$.

6. **Lemma.** For $u \in N$, $\|u\|_0 \leq c_0 \|u\|_{-1}$.

**Proof.** Using the Projection Theorem we write $u = u' + u''$, with $u' \in N$ and $u'' \in N_t = \{v \in H^0 : (u, v) = 0 \text{ for } u \in N\}$. Then

$$\|u\|_0 = \sup \{|(u, v)| : \|v\|_0 \leq 1\} = \sup \{|(u, v)| : v \in N, \|v\|_1 \leq 1\} \leq c_0 \sup \{|(u, v)| : v \in N, \|v\|_1 \leq 1\} \
\leq c_0 \sup \{|(u', v)| : \|v\|_1 \leq 1\} = c_0 \|u'\|_{-1}.$$

For $-1 \leq t \leq 1$ we let $N^t$ denote $N$ with the topology induced by $H^t$. By the preceding lemma and Theorem 3 we have

7. **Proposition.** If $-1 \leq s < t \leq 1$ the spaces $N^s$ and $N^t$ are topologically isomorphic.

Using the notation of the proof of Lemma 6 the map $P : H^0 \to N^t$ given by $u \mapsto u'$ is continuous.

8. **Proposition.** $P \in \mathcal{L}(H^t, N^s)$ for $-1 \leq t \leq 1$.

**Proof.** We let $Pu = u'$. Then for $u \in H^t$ $\|u'\|_1 \leq c_0 \|u'\|_0 \leq c_0 \|u\|_0 \leq c_0 \|u\|_1$. Now for $u \in H^0$, $\|u'\|_2 \leq \|u''\|_0 = \|u', u''\| = \|(u', u')\| \leq \|u'\|_1 \|u'\|_1 \leq c_0 \|u'\|_{-1} \|u'\|_{-1}$. Thus $\|u'\|_{-1} \leq c_0 \|u\|_{-1}$. For $-1 < t < 0$ and $0 < t < 1$, apply Theorem 3.

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* We use $(\cdot, \cdot)$ instead of $(\cdot, \cdot)_0$ for the scalar product in $H^0$. 

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Let \( N_1^t \) for \(-1 \leq t \leq 1\) be the set of \( w \in H^t \) such that for \( u \in N \), 
\((u, w) = 0\). The following two results are then simple consequences of 
the preceding facts.

9. **Theorem.** Let \( u \in H^t \), \(-1 \leq t \leq 1\). Then \( u = u' + u'' \) with \( u' \in N \) 
and \( u'' \in N_1^t \).

10. **Lemma.** If \(-1 \leq t \leq 1\) and \( v \in N_1^t \) then 
\[
\|v\|_t \leq c_t \sup \left\{ |(u, v)| : u \in N_1^{-t} \text{ and } \|u\|_{-t} \leq 1 \right\}.
\]

11. **Theorem.** For \(-1 \leq t \leq 1\) let \( f \) be a continuous linear functional 
on \( N_1^t \). Then there exists a \( v \in N_1^{-t} : f(u) = (u, v) \) for \( u \in N_1^t \).

Using the preceding results the proof is identical to that given in 
Schechter [6].

12. **Remark.** Let \( V \) be a closed subspace of \( H^t \) containing \( N \). For 
\( u \in H^0 \) let 
\[
\|u\|_{V^{-1}} = \sup \left\{ |(u, v)| : v \in V \text{ and } \|v\|_{-1} \leq 1 \right\}. 
\]
Let \( V^{-1} \) be the completion of \( H^0 \) in the norm \( \| - \|_{V^{-1}} \). Clearly \( H^{-1} \subset V^{-1} \). It is easy 
to see that \( V^{-1} \) can be identified with the dual space of \( V \) and that 
Proposition 8 is true for \( V^{-1} \).

13. **Remark.** Using Propositions 7 and 8 it is not hard to show that 
the estimates of Schechter [7] and the \( L^2 \) version of the estimates of 
Schechter [8], [9] can be obtained without assuming the finite 
dimensionality of kernel of the elliptic operator. The closure of the image 
in \( L^2 \) seems to be essential.

*Added in proof.* The \( L^2 \) version of the estimates of Schechter, Math. 
Scand. (1963), 47–69, can also be obtained from these results.

**Bibliography**


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