ORTHOGONAL EXPANSIONS WITH POSITIVE COEFFICIENTS

RICHARD ASKEY

In this note we give a simple proof of a generalization of a theorem of Szegő. The theorem in question is

\[(1) \quad (\sin \theta)^{2k-1} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n} \sin(n + 2k + 1) \theta \]

where \(\lambda > 0, \lambda \neq 1, 2, \ldots, \) and

\[\alpha_{k,n} = \frac{2^{2-k}(n + k)!\Gamma(n + 2\lambda)\Gamma(k + 1 - \lambda)}{\Gamma(\lambda)\Gamma(1 - \lambda)k!n!\Gamma(n + k + \lambda + 1)}\]

and \(P_n^{(\lambda)}(\cos \theta)\) is given by

\[(1 - 2r \cos \theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(\cos \theta)r^n.\]

Recall that \(\sin(n + 1)\theta/\sin \theta = P_n^{(1)}(\cos \theta)\) so that (1) may be written

\[(\sin \theta)^{2k-1} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n} P_{n+2k}^{(1)}(\cos \theta)(\sin \theta)\]

or

\[P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n} P_{n+2k}^{(1)}(\cos \theta)(\sin \theta)^{2k}.\]

This suggests that a formula of the form

\[(2) \quad (\sin \theta)^{2k} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n} P_n^{(\mu)}(\cos \theta)(\sin \theta)^{2k}\]

is true. Since \(\alpha_{k,n}^{(\mu)}\) is positive for \(0 < \lambda < 1\) we might conjecture that \(\alpha_{k,n}^{(\mu)}\) is positive for \(\lambda < \mu\). In fact, we show that it is for \((\mu - 1)/2 < \lambda < \mu\). The condition \((\mu - 1)/2 < \lambda\) is necessary to obtain convergence of the series (2).

This result follows from an old result of Gegenbauer [1] which has almost been forgotten by the mathematical community. Since \(P_n^{(\omega)}(x)\) is a polynomial we may write

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Gegenbauer gives the value of $\beta$ as

$$
\beta_{k,n} = \frac{\Gamma(\lambda)(k + \lambda)\Gamma((n - k)/2 + \mu - \lambda)\Gamma(\mu + (k + n)/2)}{\Gamma(\mu)((n - k)/2)!\Gamma(\mu - \lambda)\Gamma(\lambda + 1 + (k + n)/2)}
$$

if $n - k$ is even, $n \geq k$, and $\beta_{k,n} = 0$ otherwise. Observe that $\beta \geq 0$ if $\mu \geq \lambda$. A simple proof of this is in [2].

We also need the following simple fact. Let $w(x)$ and $w_1(x)$ be positive functions on a set $E$. Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be the orthonormal polynomials associated with $w(x)$ and $w_1(x)$ respectively. Then if

$$
q_n(x) = \sum_{k=0}^{n} c_{k,n} p_k(x)
$$

we have

$$
w(x)p_k(x) = \sum_{n=k}^{\infty} c_{k,n} q_n(x)w_1(x).
$$

The convergence of the series can be taken in the appropriate $L^2$ space if $[w(x)]^2/\omega_1(x)$ is integrable. (5) follows immediately from (4) since the Fourier coefficients are the same in the two expansions.

Using (3), (4) and (5) and remembering that $\{P_n^{(\alpha)}(x)\}$ are orthogonal but not orthonormal we get

$$
(1 - x^2)^{1/2} P_n^{(\alpha)}(x) = \sum_{k=0}^{\infty} \alpha_{k,n}^{(\alpha)} P_{n+2k}^{(\alpha)}(1 - x^2)^{1/2}
$$

where

$$
\alpha_{k,n}^{(\alpha)} = \frac{\Gamma(\mu)2^{2k+2\alpha}(n + 2k + \mu)(n + 2k)!}{\Gamma(\mu - \lambda)\Gamma(\mu + \lambda + 1)(n + 2k + 2\mu)}.
$$

Using the asymptotic formula for $P_n^{(\alpha)}(x)$ it is easy to see that the series in (6) converges for $-1 < x < 1$ if $\lambda > (\mu - 1)/2$. Also observe that $\alpha_{k,n}^{(\alpha)} > 0$ for $(\mu - 1)/2 < \lambda < \mu$.

Setting $n = 0$ in (6) we get

$$
(1 - x^2)^{-2\alpha} = \frac{\Gamma(\mu)2^{2\mu - 2\alpha}}{\Gamma(\alpha)\Gamma(\mu - \alpha)} \cdot \sum_{k=0}^{\infty} \frac{(2k + \mu)(2k)!\Gamma(k + \mu)\Gamma(k + \alpha)}{k!\Gamma(k + \mu - \alpha + 1)\Gamma(2k + 2\mu)} P_{2k}^{(\mu)}(x).
$$
For $\alpha = 1/2$ and $\mu = 1/2$ this is due to Bauer and $\alpha = 1/2, \mu > 0$ it is due to Gegenbauer [1].

There are two other instances of polynomial expansions of different orthogonal polynomials that involve the classical polynomials and have positive coefficients.

The better known one is

$$L_n^{(\beta)}(x) = \sum_{k=0}^{n} \frac{\Gamma(\beta - \alpha + n - k)}{\Gamma(\beta - \alpha)(n - k)!} L_k^{(\alpha)}(x),$$

where $L_n^{(\beta)}(x)$ are the Laguerre polynomials of order $\beta$, degree $n$, and $\beta > \alpha$. Notice that the coefficients are positive. See [3, p. 209] for (8).

For Jacobi polynomials, $P_n^{(\alpha, \beta)}(x)$, the following expansion holds

$$P_n^{(\alpha, \gamma)}(x) = \frac{\Gamma(n+\gamma+1)}{\Gamma(\beta-\alpha)\Gamma(n+\gamma+\beta+1)} \sum_{k=0}^{n} \frac{\Gamma(n+k+\beta+\gamma+1)\Gamma(n-k+\beta-\alpha)\Gamma(k+\alpha+\gamma+1)(2k+\alpha+\gamma+1)}{\Gamma(n-k+1)\Gamma(k+\gamma+1)} P_k^{(\alpha, \gamma)}(x).$$

See [4, p. 254]. Again notice that, if $\gamma > -1$ and $\beta \geq 0$, the coefficients are positive. The common feature of both of these results is that the polynomials are normalized correctly at the right point. In the Laguerre case $L_n^{(\alpha)}(0) > 0$ and $P_n^{(\alpha, \beta)}(1) > 0$ for Jacobi polynomials. An interesting conjecture can be formulated of which these are a special case. For convenience we formulate it on $(0, \infty)$.

Let $w(x)$ be a positive function on $(0, \infty)$ such that $\int_0^\infty x^n w(x) dx$ exists for each $n = 0, 1, \cdots$. Let $\{\rho_n(x)\}$ be the orthonormal polynomials with respect to $w(x)$ normalized by $\rho_n(0) > 0$. All of the zeros of $\rho_n(x)$ are in $(0, \infty)$ so this is possible. Let $\{\rho_n^*(x)\}$ be the polynomials orthonormal with respect to $x^n w(x)$ normalized in the same way. Then if $\alpha > 0$ we conjecture that

$$\rho_n^*(x) = \sum_{k=0}^{n} \alpha_k \rho_k(x)$$

with $\alpha_k > 0$.

For $\alpha$ an integer this follows from known classical results. For if $\alpha = 1$ then

$$c_n \rho_n^1(x) = \frac{\rho_{n+1}(0) \rho_n(x) - \rho_n(0) \rho_{n+1}(x)}{x}$$

by a theorem of Christoffel [4, Theorem 2.5]. By another theorem of Christoffel [4, Theorem 3.2.2],
Thus

\[ p_n(x) = r_n \sum_{k=0}^{n} p_k(0)p_k(x) \]

and \( p_n(0), p_n'(0) \), and thus \( r_n \) are positive by assumption. For larger integral values of \( \alpha \) the result follows by iteration.

**Bibliography**


**University of Wisconsin**