ON THE MEAN MODULUS OF TRIGONOMETRIC POLYNOMIALS WHOSE COEFFICIENTS HAVE RANDOM SIGNS

S. UCHIYAMA

1. Introduction. Let \((m_k)\) be an infinite sequence of distinct integers. We denote by \(s_n(t, x)\) \((0 < t < 1, 0 \leq x < 1)\) the \(n\)th partial sum of the trigonometric series

\[
\sum_{k=1}^{\infty} \phi_k(t)c_k\cos(m_kx),
\]

where \((\phi_k(t))\) is the system of Rademacher's functions, i.e.

\[
\phi_k(t) = \text{sign } \sin 2^k\pi t \quad (k = 1, 2, \ldots),
\]

and \(e(x) = e^{ix}\). The coefficients \(c_k\) may be complex numbers.

R. Salem and A. Zygmund [2] have investigated in detail the order of magnitude of the maximum modulus of \(s_n(t, x)\),

\[
\max_{0 \leq x < 1} |s_n(t, x)|,
\]

for the special case of \((m_k) = (k)\), the sequence of positive integers.

The purpose of this note is to present some results concerning the order of magnitude of the mean modulus

\[
\int_0^1 |s_n(t, x)| \, dx,
\]

where the sequence \((m_k)\) may be arbitrary.

It is clear that for every \(t\) and every \(n\)

\[
\int_0^1 |s_n(t, x)| \, dx \leq R_n^{1/2},
\]

where \(R_n = \sum_{k=1}^{\infty} |c_k|^2\), whatever the sequence \((m_k)\) may be. As to the lower bound for \(\int_0^1 |s_n(t, x)| \, dx\) we shall prove the following theorems:

**Theorem 1.** Let \((m_k)\) be an arbitrary sequence of distinct integers. Then, given any \(\varepsilon > 0\), there exists a positive constant \(B_\varepsilon\) depending only on \(\varepsilon\), such that except for a set of \(t\)'s of measure less than \(\varepsilon\) we have

Received by the editors December 14, 1964.
We write for the sake of brevity

\[ S_n = \sum |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2, \]

the summation being extended over all indices i, j, k, l with 1 \( \leq i, j, k, l \leq n \) such that \( m_i + m_j = m_k + m_l \) (order is relevant).

**Theorem 2.** Let \((m_k)\) be an arbitrary sequence of distinct integers. If \( S_n / R_n^a = O(n^{-b}) \) for some \( a > 1 \), then we have

\[ \lim \inf_{n \to \infty} R_n^{-1/2} \int_0^1 |s_n(t, x)| \, dx \geq 2^{-1/2} \]

almost everywhere in t.

The condition imposed on \( S_n / R_n^a \) in Theorem 2 obviously implies that \( R_n \) tends to infinity with \( n \). Note that for any sequence \((m_k)\) of distinct integers we have always \( 1 \leq \alpha \leq 2 \), if \( S_n / R_n^b = O(n^{-c}) \); in the case of \( \alpha = 1 \), which is excluded from Theorem 2, one may also prove the validity of (2), assuming that the growth of \( R_n \) as \( n \to \infty \) is sufficiently regular (see §3 below).

**2. Proof of the theorems.** We have

\[ \int_0^1 |s_n(t, x)|^2 \, dx = \sum_{k=1}^n |\phi_k(t)|^2 \leq R_n \]

almost everywhere in t. Hence we obtain by Hölder’s inequality

\[ R_n = \int_0^1 |s_n(t, x)|^2 \, dx \leq \left( \int_0^1 |s_n(t, x)| \, dx \right)^{2/3} \left( \int_0^1 |s_n(t, x)|^4 \, dx \right)^{1/3} \]

for almost all t, where, as is readily seen,

\[ \int_0^1 dt \int_0^1 |s_n(t, x)|^4 \, dx = 2R_n^2 - T_n, \quad T_n = \sum_{k=1}^n |c_k|^4. \]

Let \( E \) denote the set of t, \( 0 < t < 1 \), for which the integral
\[ \int_0^1 |s_n(t, x)|^4 \, dx \geq 2R_n^2 - T_n + A. \]

Then it follows from (4) that the measure \( m(E) \) of \( E \) satisfies
\[ m(E) \leq \frac{2R_n^2 - T_n}{2R_n^2 - T_n + A} \leq \frac{2}{2 + a} < \epsilon, \]
if we put \( A = aR_n^2 \) and take \( a > 0 \) sufficiently large. Thus, for any \( t \in E \) we have
\[ \int_0^1 |s_n(t, x)|^4 \, dx < (2 + a)R_n^2, \]
and we obtain, via (3),
\[ \int_0^1 |s_n(t, x)|^4 \, dx \geq \frac{R_n^2}{(2 + a)^{1/2} R_n} = B_n R_n^{1/2} \]
with \( B_n = (2 + a)^{-1/2} \). This is (1).

Now, let us consider the integral
\[ I_n = \int_0^1 R_n^{-4} \left( \int_0^1 |s_n(t, x)|^4 \, dx - 2R_n^2 + T_n \right)^2 \, dt. \]

It is not difficult to verify that
\[ \int_0^1 \left( \int_0^1 |s_n(t, x)|^4 \, dx \right)^2 \, dt = (2R_n^2 - T_n)^2 + O(S_n). \]

By the assumption \( S_n/R_n^4 = O(n^{-a}) \) (\( a > 1 \)) and the relation (4) we thus have \( I_n = O(n^{-a}) \) and therefore \( \sum I_n < \infty \). Hence
\[ \sum_{n=1}^\infty R_n^{-4} \left( \int_0^1 |s_n(t, x)|^4 \, dx - 2R_n^2 + T_n \right)^2 < \infty \]
for almost all \( t \). It follows in particular that for almost all \( t \) and any \( \epsilon > 0 \) we have for all \( n \geq n_0(t, \epsilon) \)
\[ \int_0^1 |s_n(t, x)|^4 \, dx < (2 + \epsilon)R_n^2 \]
so that
by (3) again. Since $\epsilon>0$ is arbitrary, this proves (2).

3. Remarks. (1) Let us consider the case of $S_n/R^n=O(1/n)$. It will be immediately clear from the argument of §2 that we have

$$\sum_{n=1}^\infty I_{m^*} < \infty,$$

so that

$$\liminf_{m \to \infty} R_n^{-1/2} \int_0^1 |s_{m^*}(t, x)| \, dx \geq 2^{-1/2}$$

for almost all $t$.

Suppose now that

$$R_{(m+1)^3}/R_m^3 \to 1 \quad \text{as} \quad m \to \infty.$$ Let $n$ be any integer between $n^2$ and $(n+1)^2$. Then from the inequality

$$\left| \int_0^1 s_n(t, x) \, dx - \int_0^1 s_{m^*}(t, x) \, dx \right| \leq \int_0^1 \left| s_n(t, x) - s_{m^*}(t, x) \right| \, dx \leq (R_n - R_m)^{1/2}$$

it follows that

$$R_n^{-1/2} \int_0^1 |s_n(t, x)| \, dx \geq \left( \frac{R_{(m+1)^3}}{R_m^3} \right)^{-1/2} R_m^{-1/2} \int_0^1 |s_{m^*}(t, x)| \, dx \quad \text{for any sequence } \{m_k\} \text{ of distinct integers.}$$

Hence

$$\liminf_{n \to \infty} R_n^{-1/2} \int_0^1 |s_n(t, x)| \, dx \geq 2^{-1/2}$$

for almost all $t$.

As an example we take $c_k=1 \ (k=1, 2, \ldots)$. We have then $R_n=R_n=n$, and $S_n=O(n^3)$ for any sequence $(m_k)$ of distinct integers. Therefore
\[
\liminf_{n \to \infty} n^{-1/2} \int_0^1 \left| \sum_{k=1}^n \phi_k(t) e(m_kx) \right| dx \geq 2^{-1/2}
\]
almost everywhere in \(t\).

(2) It is obvious that our Theorems 1 and 2 have analogues for partial sums \(t_n(t, x)\) of real trigonometric series
\[
\sum_{k=1}^\infty \phi_k(t) a_k \cos 2\pi(m_kx - \alpha_k)
\]
with real coefficients \(a_k\) and phases \(\alpha_k\). Put
\[
P_n = \frac{1}{2} \sum_{k=1}^n a_k^2
\]
and
\[
Q_n = \sum_{1 \leq i, j, k, l \leq n} a_i a_j a_k a_l,
\]
where the summation is taken over \(1 \leq i, j, k, l \leq n\) such that
\[
\pm m_i \pm m_j \pm m_k \pm m_l = 0.
\]
Then, given any \(\epsilon > 0\), we have for all \(t\) but a set of measure less than \(\epsilon\) and all \(n \geq 1\)
\[
\int_0^1 |t_n(t, x)| dx \geq C_\epsilon P_n^{1/2}
\]
with some constant \(C_\epsilon > 0\), and, if \(\beta > 1\) or \(\beta = 1\) and \(P_{(m+1)^2}/P_m^2 \to 1\) as \(m \to \infty\), where \(Q_n/P_n^4 = O(n^{-\beta})\), we have for almost all \(t\)
\[
\liminf_{n \to \infty} P_n^{-1/2} \int_0^1 |t_n(t, x)| dx \geq 3^{-1/2}.
\]
The proof of these results is quite similar to that of the results for complex polynomials \(s_n(t, x)\).

(3) Let \(m_1, \ldots, m_n\) be any set of \(n\) distinct integers. S. Chowla has conjectured that
\[
\min_{0 \leq x < 1} \sum_{k=1}^n \cos 2\pi m_kx < - Cn^{1/2}
\]
for some absolute constant \(C > 0\) (cf. [1]). If this conjecture is true, it is essentially the best possible.
We can show that, given \( n \) distinct integers \( m_1, \ldots, m_n \), there exists always a subset \( m_{i_1}, \ldots, m_{i_r} \) of \( m_1, \ldots, m_n \) for which
\[
\min_{0 \leq x < 1} \sum_{j=1}^{r} \cos 2\pi m_{i_j} x < -\frac{1}{4} \left( \frac{n}{6} \right)^{1/3}.
\]
This is an easy consequence of the fact that
\[
\int_0^1 \left| \sum_{k=1}^{n} \varepsilon_k \cos 2\pi m_k x \right| \, dx > \left( \frac{n}{6} \right)^{1/2}
\]
for some sequence \((\varepsilon_k)\) of \( \pm 1 \), which is a particular case of the results mentioned in (2).

**References**