ON THE MEAN MODULUS OF TRIGONOMETRIC POLYNOMIALS WHOSE COEFFICIENTS HAVE RANDOM SIGNS

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1. Introduction. Let \((m_k)\) be an infinite sequence of distinct integers. We denote by \(s_n(t, x)\) \((0 < t < 1, 0 \leq x < 1)\) the \(n\)th partial sum of the trigonometric series

\[
\sum_{k=1}^{\infty} \phi_k(t)c_k e(m_kx),
\]

where \((\phi_k(t))\) is the system of Rademacher's functions, i.e.

\[
\phi_k(t) = \text{sign} \sin 2^k \pi t \quad (k = 1, 2, \ldots),
\]

and \(e(x) = e^{2\pi i x}\). The coefficients \(c_k\) may be complex numbers.

R. Salem and A. Zygmund \([2]\) have investigated in detail the order of magnitude of the maximum modulus of \(s_n(t, x)\),

\[
\max_{0 \leq x < 1} |s_n(t, x)|,
\]

for the special case of \((m_k) = (k)\), the sequence of positive integers. The purpose of this note is to present some results concerning the order of magnitude of the mean modulus

\[
\int_0^1 |s_n(t, x)| \, dx,
\]

where the sequence \((m_k)\) may be arbitrary.

It is clear that for every \(t\) and every \(n\)

\[
\int_0^1 |s_n(t, x)| \, dx \leq R_n^{1/2},
\]

where \(R_n = \sum_{k=1}^{\infty} |c_k|^2\), whatever the sequence \((m_k)\) may be. As to the lower bound for \(\int_0^1 |s_n(t, x)| \, dx\) we shall prove the following theorems:

**Theorem 1.** Let \((m_k)\) be an arbitrary sequence of distinct integers. Then, given any \(\varepsilon > 0\), there exists a positive constant \(B_\varepsilon\) depending only on \(\varepsilon\), such that except for a set of \(t\)s of measure less than \(\varepsilon\) we have

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for all \( n \geq 1 \).

We write for the sake of brevity

\[
S_n = \sum |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2,
\]

the summation being extended over all indices \( i, j, k, l \) with \( 1 \leq i, j, k, l \leq n \) such that \( m_i + m_j = m_k + m_l \) (order is relevant).

**Theorem 2.** Let \((m_k)\) be an arbitrary sequence of distinct integers. If \( S_n/R_n^a = O(n^{-\alpha}) \) for some \( \alpha > 1 \), then we have

\[
\liminf_{n \to \infty} R_n^{-1/2} \int_0^1 |s_n(t, x)|^2 dx \geq 2^{-1/2}
\]

almost everywhere in \( t \).

The condition imposed on \( S_n/R_n^a \) in Theorem 2 obviously implies that \( R_n \) tends to infinity with \( n \). Note that for any sequence \((m_k)\) of distinct integers we have always \( 1 \leq \alpha \leq 2 \), if \( S_n/R_n^a = O(n^{-\alpha}) \); in the case of \( \alpha = 1 \), which is excluded from Theorem 2, one may also prove the validity of (2), assuming that the growth of \( R_n \) as \( n \to \infty \) is sufficiently regular (see §3 below).

2. Proof of the theorems. We have

\[
\int_0^1 |s_n(t, x)|^2 dx = \sum_{k=1}^n \phi_k(t) |c_k|^2 = R_n
\]

almost everywhere in \( t \). Hence we obtain by Hölder's inequality

\[
R_n = \int_0^1 |s_n(t, x)|^2 dx
\]

\[
\leq \left( \int_0^1 |s_n(t, x)|^2 dx \right)^{2/3} \left( \int_0^1 |s_n(t, x)|^4 dx \right)^{1/3}
\]

for almost all \( t \), where, as is readily seen,

\[
\int_0^1 dt \int_0^1 |s_n(t, x)|^4 dx = 2R_n^2 - T_n, \quad T_n = \sum_{k=1}^n |c_k|^4.
\]

Let \( E \) denote the set of \( t, 0 < t < 1 \), for which the integral
\[\int_0^1 |s_n(t, x)|^4 dx \geq 2R_n^2 - T_n + A.\]

Then it follows from (4) that the measure \(m(E)\) of \(E\) satisfies

\[m(E) \leq \frac{2R_n^2 - T_n}{2R_n^2 - T_n + A} \leq \frac{2}{2 + a} < \epsilon,\]

if we put \(A = aR_n^a\) and take \(a > 0\) sufficiently large. Thus, for any \(t \in I\) we have

\[\int_0^1 |s_n(t, x)|^4 dx < (2 + a)R_n^2,\]

and we obtain, via (3),

\[\int_0^1 |s_n(t, x)|^4 dx \geq \frac{R_n^{3/2}}{(2 + a)^{1/2}R_n} = B_nR_n^{1/3}\]

with \(B_n = (2 + a)^{-1/2}\). This is (1).

Now, let us consider the integral

\[I_n = \int_0^1 R_n^{-4} \left( \int_0^1 |s_n(t, x)|^4 dx - 2R_n^2 + T_n \right)^2 dt.\]

It is not difficult to verify that

\[\int_0^1 \left( \int_0^1 |s_n(t, x)|^4 dx \right)^2 dt = (2R_n^2 - T_n)^2 + O(S_n).\]

By the assumption \(S_n/R_n^a = O(n^{-a}) (a > 1)\) and the relation (4) we thus have \(I_n = O(n^{-a})\) and therefore \(\sum I_n < \infty\). Hence

\[\sum_{n=1}^\infty R_n^{-4} \left( \int_0^1 |s_n(t, x)|^4 dx - 2R_n^2 + T_n \right)^2 < \infty\]

for almost all \(t\). It follows in particular that for almost all \(t\) and any \(\epsilon > 0\) we have for all \(n \geq n_0(t, \epsilon)\)

\[\int_0^1 |s_n(t, x)|^4 dx < (2 + \epsilon)R_n^2,\]

so that
$R_n^{-1/2} \int_0^1 |s_n(t, x)| \, dx > (2 + \varepsilon)^{-1/2}$

by (3) again. Since $\varepsilon > 0$ is arbitrary, this proves (2).

3. Remarks. (1) Let us consider the case of $S_n/R_n^a = O(1/n)$. It will be immediately clear from the argument of §2 that we have

$$\sum_{n=1}^{\infty} I_{n^a} < \infty,$$

so that

$$\liminf_{m \to \infty} R_n^{-1/2} \int_0^1 |s_{n^a}(t, x)| \, dx \geq 2^{-1/2}$$

for almost all $t$.

Suppose now that

$$\frac{R_{(m+1)^a}}{R_m^a} \to 1 \quad \text{as} \quad m \to \infty.$$

Let $n$ be any integer between $m^2$ and $(m+1)^2$. Then from the inequality

$$\left| \int_0^1 |s_n(t, x)| \, dx - \int_0^1 |s_{n^a}(t, x)| \, dx \right| \leq \int_0^1 |s_n(t, x) - s_{n^a}(t, x)| \, dx$$

it follows that

$$R_n^{-1/2} \int_0^1 |s_n(t, x)| \, dx \geq \left( \frac{R_{(m+1)^a}}{R_m^a} \right)^{-1/2} R_m^{-1/2} \int_0^1 |s_{m^a}(t, x)| \, dx$$

$$- \left( \frac{R_{(m+1)^a}}{R_m^a} - 1 \right)^{1/2}.$$

Hence

$$\liminf_{n \to \infty} R_n^{-1/2} \int_0^1 |s_n(t, x)| \, dx \geq 2^{-1/2}$$

for almost all $t$.

As an example we take $c_k = 1 \, (k = 1, 2, \ldots)$. We have then $R_n = T_n = n$, and $S_n = O(n^a)$ for any sequence $(m_k)$ of distinct integers. Therefore
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\[
\liminf_{n \to \infty} n^{-1/2} \int_0^1 \left| \sum_{k=1}^n \phi_k(t) e(m_k x) \right| \, dx \geq 2^{-1/2}
\]

almost everywhere in \( t \).

(2) It is obvious that our Theorems 1 and 2 have analogues for partial sums \( t_n(t, x) \) of real trigonometric series

\[
\sum_{k=1}^\infty \phi_k(t) a_k \cos 2\pi (m_k x - \alpha_k)
\]

with real coefficients \( a_k \) and phases \( \alpha_k \). Put

\[
P_n = \frac{1}{2} \sum_{k=1}^n a_k^2
\]

and

\[
Q_n = \sum_{1 \leq i \neq j, k, l \leq n} a_i a_j a_k a_l
\]

where the summation is taken over \( 1 \leq i, j, k, l \leq n \) such that

\[
\pm m_i \pm m_j \pm m_k \pm m_l = 0.
\]

Then, given any \( \varepsilon > 0 \), we have for all \( t \) but a set of measure less than \( \varepsilon \) and all \( n \geq 1 \)

\[
\int_0^1 \left| t_n(t, x) \right| \, dx \geq C \varepsilon P_n^{1/2}
\]

with some constant \( C > 0 \), and, if \( \beta > 1 \) or \( \beta = 1 \) and \( P_{(m+1)^{3/2}}/P_m^{3/2} \to 1 \) as \( m \to \infty \), where \( Q_n/P_n = O(n^{-\beta}) \), we have for almost all \( t \)

\[
\liminf_{n \to \infty} P_n^{-1/2} \int_0^1 \left| t_n(t, x) \right| \, dx \geq 3^{-1/2}.
\]

The proof of these results is quite similar to that of the results for complex polynomials \( s_n(t, x) \).

(3) Let \( m_1, \ldots, m_n \) be any set of \( n \) distinct integers. S. Chowla has conjectured that

\[
\min_{0 \leq x < 1} \sum_{k=1}^n \cos 2\pi m_k x < - C n^{1/2}
\]

for some absolute constant \( C > 0 \) (cf. [1]). If this conjecture is true, it is essentially the best possible.
We can show that, given \( n \) distinct integers \( m_1, \ldots, m_n \), there exists always a subset \( m_{i_1}, \ldots, m_{i_r} \) of \( m_1, \ldots, m_n \) for which

\[
\min_{0 \leq x < 1} \sum_{j=1}^{r} \cos 2\pi m_{i_j}x < -\frac{1}{4} \left( \frac{n}{6} \right)^{1/2}.
\]

This is an easy consequence of the fact that

\[
\int_{0}^{1} \left| \sum_{k=1}^{n} a_k \cos 2\pi m_kx \right| dx > \left( \frac{n}{6} \right)^{1/2}
\]

for some sequence \( (a_k) \) of \( \pm 1 \), which is a particular case of the results mentioned in (2).

References


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