LOCAL CONVEXITY AND $L_n$ SETS

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We will prove that a closed connected set $S$ in $E_r$ whose points of local nonconvexity can be decomposed into a finite number of convex parts also possesses a simple type of polygonal connectedness. Furthermore, if $S$ has a finite number of points of local nonconvexity, we obtain another result. In order to describe this situation easily we use the notation of $L_n$ set used by Horn and Valentine [1] in 1949. For other results related to $L_n$ sets, see A. Bruckner and J. Bruckner [1] and Valentine [1]. The following notations and definitions will enable us to express the results more readily.

Notations. The interior, boundary and closure of a set $S$ in Euclidean $r$-space $E_r$ are denoted by int $S$, bd $S$ and cl $S$ respectively. If $x \in S$, $y \in S$, then $xy$ denotes the closed line segment joining $x$ and $y$. The symbols $\cup$, $\cap$ and $\sim$ denote set union, set intersection and set difference respectively. The symbol $\emptyset$ is used to denote the empty set. The convex hull of a set $S \subseteq E_r$ is indicated by conv $S$.

Definition 1. A set $S$ is called an $L_n$ set if each pair of points in $S$ can be joined by a polygonal arc of $S$ containing at most $n$ segments.

Definition 2. A point $x \in S$ is a point of local convexity of $S$ if there exists a neighborhood $N$ of $x$ such that $N \cap S$ is convex; otherwise, $x$ is called a point of local nonconvexity of $S$.

Definition 3. A set $S$ is starshaped relative to a point $p$ if for every $x \in S$ it is true that $px \subseteq S$.

The following two theorems together with Theorem 3 stated later contain the main results of this paper. The symbols $n$ and $r$ always denote non-negative integers.

Theorem 1. Suppose $S$ is a closed connected set in $E_r$ which has at most $n$ points of local nonconvexity.

Then $S$ is an $L_{n+1}$ set.

Theorem 2. Suppose $S$ is a closed connected set in $E_r$. Furthermore, suppose that the set $Q$ of all points of local nonconvexity of $S$ can be decomposed into $n$ convex subsets.

Then $S$ is an $L_{2n+1}$ set.

It should be observed that if $n=0$ in Theorems 1 and 2, then each

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reduces to the important theorem of Tietze [1] which we state as follows, since it will be used.

**Theorem (Tietze [1]).** If $S$ is a closed connected set in $E_r$ all of whose points are points of local convexity, then $S$ is convex.

**Corollary 1 (to Tietze's Theorem).** Let $S$ be a closed set in $E_r$. Suppose $x \in S$, $y \in S$, $z \in S$ with $xy \subseteq S$, $yz \subseteq S$. If the triangular set $\Delta = \text{conv}(x \cup y \cup z)$ determined by $x$, $y$, $z$ contains no points of local nonconvexity of $S$, then $\Delta \subseteq S$.

**Proof.** Corollary 1 follows from Tietze's Theorem since that component of $S \cap \Delta$ which contains $xy$ and $yz$ is a closed connected set having no points of local nonconvexity.

**Corollary 2 (to Corollary 1).** Let $S$ be a closed set in $E_r$. Suppose $x \in S$, $y \in S$ with $xy \subseteq S$, $yz \subseteq S$. If the only possible points of local nonconvexity of $S$ in $\Delta = \text{conv}(x \cup y \cup z)$ are $x$ and $z$, then $\Delta \subseteq S$.

**Proof.** Choose points $x_i \in xy$, $z_i \in yz$ so that $\text{conv}(x_i \cup y \cup z_i)$ contains no points of local nonconvexity of $S$. Corollary 1 implies $\text{conv}(x_i \cup y \cup z_i) \subseteq S$. Since $S$ is closed, an obvious limit procedure implies $\Delta \subseteq S$.

The following lemma will shorten the proofs of Theorems 1 and 2.

**Lemma 1.** Suppose $S$ is a closed connected set which has at least one point of local nonconvexity. If $x$ is a point of $S$, then there exists at least one point $p$ of local nonconvexity of $S$ such that $xp \subseteq S$.

**Proof.** Let $Q$ denote the collection of all points of local nonconvexity of $S$. Since $S$ is closed, Definition 2 implies that $Q$ is closed. Firstly, if $x \in Q$, Lemma 1 follows by choosing $p = x$, since $xp \subseteq S$.

Secondly, if $x \notin Q$, let $S(x)$ denote the set of all points of $S$ which can see $x$ via $S$, so that

$$S(x) = \{y : xy \subseteq S\}.$$  

We wish to prove that $Q \cap S(x) \neq \emptyset$. To do this, suppose $Q \cap S(x) = \emptyset$. We will prove that this implies $S(x)$ is both open and closed in $S$, so that $S(x) = S$, $S$ being connected. To prove $S(x)$ is open relative to $S$, choose $y \in S(x)$. Since $Q$ is closed, condition (1) and $Q \cap S(x) = \emptyset$ imply there exists a neighborhood $N(y)$ of $y$ (with $S \cap N(y)$ convex) such that if $z \in S \cap N(y)$, then $Q \cap \text{conv}(x \cup y \cup z) = \emptyset$. Corollary 1 then implies that $xz \subseteq S$. Hence, $S \cap N(y) \subseteq S(x)$ so that $S(x)$ is open relative to $S$. To prove that $S$ is also closed relative to $S$ let $y_j \rightarrow y$ ($j = 1, 2, \cdots$) where $y_j \in S(x)$. Then, since $xy_j \subseteq S(x)$, and since $S$ is closed, we have $xy \subseteq S$ so that $y \in S(x)$. Hence $S(x) = S$, since $S$ is...
connected. This and the assumption \( Q \cap S(x) = 0 \) imply \( S \cap Q = 0 \).
Since \( S \neq 0 \), we have \( Q = 0 \), a contradiction, because the cardinality of \( Q \) is greater than zero, by assumption. Thus, \( Q \cap S(x) \neq 0 \), and Lemma 1 has been proved.

Finally, before proving Theorems 1 and 2, we state the following useful concept.

**Definition 4.** Let \( Q \) be a subset of a set \( S \) in \( E_r \). An \( s \)-fold connected convex covering \( \mathcal{P} \) for \( Q \) is a collection of \( s \) or fewer closed convex sets whose union is a connected subset of \( S \) which contains \( Q \).

**Proof of Theorem 2.** First, observe that if \( n = 0 \), then Tietze's Theorem [1] implies Theorem 2. Hence, suppose \( n \geq 1 \). Our hypotheses imply that the collection \( Q \) of all points of local nonconvexity can be decomposed into \( n \) closed convex subsets which we designate by \( Q_1, \ldots, Q_n \), so that

\[
Q = \bigcup_{i=1}^{n} Q_i.
\]

The sets \( Q_i \) \((i = 1, \ldots, n)\) may be taken to be closed and convex since \( S \) is closed. They may or may not be disjoint. We will prove that there exists a \((2n - 1)\)-fold connected convex covering \( \mathcal{P} \) for \( Q \) as described in Definition 4. If \( n = 1 \), let \( \mathcal{P} = \{Q\} \), since \( Q \) is a 1-fold connected convex covering of itself. Hence, suppose \( n > 1 \). Since each set \( Q_1 \) is its own 1-fold connected convex covering, the collection \( \{Q_1, Q_2, \ldots, Q_n\} \) has a maximal subcollection of \( s \) members, which, relative to \( S \), has a \((2s - 1)\)-fold connected convex covering. Without loss of generality we will rearrange subscripts so that this maximal subcollection is

\[
\mathcal{P}_1 = \{Q_1, Q_2, \ldots, Q_s\}, \quad s \leq n.
\]

Remember that \( \bigcup_{i=1}^{s} Q_i \) has a \((2s - 1)\)-fold connected convex covering in \( S \). We will prove that \( s = n \). To do this suppose \( s < n \), and let

\[
\mathcal{P}_2 = \{Q_{s+1}, Q_{s+2}, \ldots, Q_n\}.
\]

Also let

\[
Q^1 = \bigcup_{i=1}^{s} Q_i, \quad Q^2 = \bigcup_{i=s+1}^{n} Q_i.
\]

The maximality of \( \mathcal{P}_1 \) as described above implies \( Q^1 \cap Q^2 = 0 \). Lemma 1 implies that for each point \( x \in S \) there exists either a point \( y_1 \in Q^1 \) or a point \( y_2 \in Q^2 \) such that at least one of the two segments \( xy_1, xy_2 \) is in \( S \). Since \( S \) is a closed connected set, the preceding sentence im-
plies the existence of at least one point \( y \in S \) and a pair of points \( y_1 \in Q^1, y_2 \in Q^2 \) such that \( y_i \ (i = 1, 2) \) is a nearest point of \( Q^i \) which satisfies the condition

\[
(3) \quad y y_i \subseteq S \quad (i = 1, 2).
\]

Hence, \(| y y_i | \) = the distance between \( y \) and \( S(y) \cap Q^i \). Let \( Q_1 \) and \( Q_k \) denote elements of \( Q_1 \) and \( Q_2 \) respectively such that

\[
(4) \quad y_1 \in Q_1, \quad y_2 \in Q_k.
\]

First, if \( y \in Q^1 \) then \( Q^1 \cup Q_k \) would have a \([2(s+1) - 1]\)-fold connected convex covering consisting of the \((2s - 1)\)-fold connected convex covering for \( Q_1 \) together with \( y_1 y_2 \) and \( Q_k \). Hence \( Q_1 \) would not be maximal, so that \( y \notin Q^1 \). In exactly the same way we have \( y \notin Q^2 \). Hence, \( y \in Q \). Now, since \( y_i \) is a nearest point of \( Q^i \) to \( y \) which satisfies (3), the maximality of \( Q_1 \) implies

\[
(5) \quad Q \cap y y_i = y_i \quad (i = 1, 2).
\]

Without loss of generality we may assume that

\[
(6) \quad | y y_1 | \geq | y y_2 |.
\]

Since \( Q^2 \cap y y_1 = 0 \), let \( z \in y y_2 \) be that point such that \( y z \in Q^2 \not= 0 \), and such that \( \text{conv}(y \cup y_1 \cup z) \) contains no points of \( Q^2 \) in its interior. Since \( y \in Q \), no points of \( Q^1 \) are in the \( \text{conv}(y \cup y_1 \cup z) \), otherwise there will exist a point \( y^* \in S(y) \cap Q^1 \) with \(| y y^* | < | y y_1 | \), thus violating (3). Let \( y_3 \) be the point of \( Q^2 \cap y y_1 \) nearest to \( y_1 \). Corollary 2 applied to \( y_1, y_3 \) and \( y \) implies \( y_1 y_3 \subseteq S \). Let \( Q_m \) be a member of \( Q_2 \) containing \( y_3 \). Since \( y_1 y_3 \subseteq S \), the collection \( Q^i \) together with \( Q_m \) would have a \((2s+1)\)-fold connected convex covering which violates the maximality of \( Q_1 \). Hence, we have a contradiction, and hence \( Q \) has a \((2m - 1)\)-fold connected convex covering.

To complete the proof, choose any pair of points \( x \) and \( y \) in \( S \). By Lemma 1 there exists two points \( x_1 \) and \( x_2 \) of \( Q \) such that \( x x_1 \subseteq S \), \( y x_2 \subseteq S \). Since \( Q \) has a \((2n - 1)\)-fold connected convex covering \( x_1 \) and \( x_2 \) can be joined by a polygonal path in \( S \) containing at most \( 2n - 1 \) segments. Therefore, \( x \) and \( y \) can be joined by a polygonal path in \( S \) containing at most \( 2n + 1 \) segments, so that \( S \) is an \( L_{2n+1} \) set.

**Proof of Theorem 1.** The proof of Theorem 1 has the same form as that of Theorem 2 with the following difference. The set \( Q \) of points of local nonconvexity of \( S \) has an \((n - 1)\)-fold connected convex covering, and the proof of this fact can be constructed as in the proof of Theorem 2. It should be observed that in this construction the convex covering is the union of line segments with endpoints in \( Q \) if
$n > 1$ and a single point if $n = 1$. The fact that $S$ is an $L_{n+1}$ set then follows immediately from Lemma 1.

Theorem 2 can be generalized as follows. We state this result last because of its more complicated structure.

**Theorem 3.** Let $S$ be a closed connected set in $E_r$. Suppose the set $Q$ of all points of local nonconvexity of $S$ can be decomposed into $n$ connected closed subsets $Q_1, Q_2, \ldots, Q_n$ such that if $x \in Q_i$, $y \in Q_i$, $Q_i \in \{Q_1, Q_2, \ldots, Q_n\}$, then $xy \subseteq S$.

Then $S$ is an $L_{2n+1}$ set.

**Proof.** Just as in the proof of Theorem 2 it can be proved that the set $Q$ can be covered by a connected set which is the union of $2n-1$ subsets $P_1, P_2, \ldots, P_{2n-1}$ such that if $x \in P_i$, $y \in P_i$, $P_i \in \{P_1, P_2, \ldots, P_{2n-1}\}$, then $xy \subseteq S$. Lemma 1 again implies that $S$ is an $L_{2n+1}$ set.

**Remarks.** It is a relatively simple matter to construct examples in $E_3$ which satisfy the hypotheses of Theorem 2 and which are not $L_{2n}$ sets, so that the theorem for $r > 2$ in one respect is a best theorem. One such example consists of the appropriate union of polyhedra in $E_3$ each of whose points of local nonconvexity consists of a single segment, together with a suitable additional number of line segments intersecting each of these polyhedra in a unique point, the latter being a point of local nonconvexity. Since the converse of Theorem 2 is false, the theorem does not characterize $L_{2n+1}$ sets. Corresponding remarks also apply to Theorem 1. Furthermore, it should be observed that Theorem 2 is of chief interest in $E_r$ with $r \geq 3$, because for sets $S$ in the plane $E_2$ the hypotheses of Theorem 2 imply the hypotheses of Theorem 1, so that $S$ is an $L_{n+1}$ set, a stronger result. However, for $r > 2$ this no longer holds, and the conclusion of Theorem 2 cannot be improved.

**Questions.** It appears that Theorem 2 can be improved if $S$ is a polyhedron or if $S$ is the closure of an open connected set. In fact the classification of points of local nonconvexity for polyhedra is itself of independent interest. In $E_2$, the set $Q$ for a simple closed polygonal region are the points at which the polygonal boundary has re-entrant angles, and there are only a finite number of these so that Theorem 1 applies. Also it is a simple matter to construct polygonal regions in $E_2$ having exactly $n$ points of local nonconvexity which are not $L_n$ sets, so that Theorem 1 is a best theorem in one sense. The corresponding result for polyhedra in $E_3$ is still unsettled, since the points of local nonconvexity for a polyhedron may consist of a complicated combination of line segments. Both Theorems 2 and 3 do shed a good deal
of light on this more complicated situation. However, there is still
more work to be done for polyhedra in $E_r, r \geq 3$. As a final observa-
tion note that if there exists a point $p \subseteq S$ such that for each $y \subseteq Q$ we
have $py \subseteq S$, then Lemma 1 implies that $S$ is an $L_4$ set. It appears
that this result might have further generalizations.

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