ON THE CHARACTERISTIC ROOTS OF MATRICES

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Let \( A = (a_{ij}) \) be an \( n \times n \) complex matrix. For \( i = 1, 2, \ldots, n \) and \( p > 0 \) define \( R(i, p) = (\sum_{j=1}^{n} |a_{ij}|^p)^{1/p} \). It is well known [2] that the characteristic roots of \( A \) lie in the union of the disks

\[
|a_{ii} - z| \leq R(i, 1) \quad (i = 1, 2, \ldots, n).
\]

A. Brauer [3] improved this result by replacing these disks with the \( n(n-1)/2 \) ovals of Cassini

\[
|a_{ii} - z| |a_{jj} - z| \leq R(i, 1)R(j, 1) \quad (i, j = 1, 2, \ldots, n; i \neq j).
\]

A. Ostrowski [4] extended (1) as follows:

THEOREM A. Let \( k_1, k_2, \ldots, k_n \) be positive numbers such that

\[
\sum_{i=1}^{n} (k_i + 1)^{-1} \leq 1.
\]

Let \( p \) and \( q \) be chosen so that

\[
1/p + 1/q = 1 \quad \text{and} \quad p, q > 1.
\]

Then the characteristic roots of \( A \) lie in the union of the disks

\[
|a_{ii} - z| \leq k^1/q R(i, p) \quad (i = 1, 2, \ldots, n).
\]

In this paper we improve Theorem A in the same way that Brauer improved (1). It is also shown that our result generalizes Brauer's theorem (2).

THEOREM. Let \( A = (a_{ij}) \) be an \( n \times n \) complex matrix. Let \( k_1, k_2, \ldots, k_n, p \) and \( q \) be positive numbers satisfying (3) and (4). Then the characteristic roots of \( A \) lie in the union of the \( n(n-1)/2 \) ovals of Cassini

\[
|a_{ii} - z| |a_{jj} - z| \leq (k_1k_2)^{1/q}R(i, p)R(j, p) \quad (i, j = 1, 2, \ldots, n; i \neq j).
\]

We first prove the following lemma:
Lemma. Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be non-negative numbers such that
\[
\sum_{i=1}^{n} b_i \leq \sum_{i=1}^{n} a_i = 1.
\]
Then there exist two integers \( u \) and \( v \) such that
\[
a_u a_v (1 - b_u - b_v) \geq b_u b_v (1 - a_u - a_v).
\]

Proof. By (7) we may pick \( u \) so that \( b_u \leq a_u \). If the lemma is false, we may assume that
\[
a_u a_j (1 - b_u - b_j) < b_u b_j (1 - a_u - a_j) \quad (j = 1, 2, \ldots, n; j \neq u).
\]
Hence
\[
a_j (1 - b_u) < b_j (1 - a_u) \quad (j = 1, 2, \ldots, n; j \neq u).
\]
By (7) we have
\[
a_j \sum_{l \neq u} b_l < b_j \sum_{l \neq u} a_l \quad (j = 1, 2, \ldots, n; j \neq u).
\]
Adding these inequalities, we obtain the contradiction
\[
\sum_{j \neq u} a_j \sum_{l \neq u} b_l < \sum_{j \neq u} b_j \sum_{l \neq u} a_l.
\]

Proof of the theorem. Let \( \theta \) be a characteristic root of \( A \), with a corresponding characteristic vector \( (x_1, x_2, \ldots, x_n) \) normalized so that
\[
\sum_{i=1}^{n} |x_i|^2 = 1.
\]
From the equations
\[
\sum_{j=1}^{n} (a_{ij} - \theta b_{ij}) x_j = 0 \quad (i = 1, 2, \ldots, n)
\]
(where \( \delta \) is the Kronecker symbol), it follows that
\[
|a_{ii} - \theta| |x_i| \leq \sum_{j \neq i; j \neq i}^{n} |a_{ij}| |x_i| \quad (i = 1, 2, \ldots, n).
\]
Hence
Applying Hölder's inequality \cite[1, p. 19]{1} and equation (8), we obtain

\begin{align}
|a_{ii} - \theta| |a_{jj} - \theta| & \leq \sum_{l=1; l \neq i}^{n} |a_{li}| |x_l| \sum_{m=1; m \neq j}^{n} |a_{mj}| |x_m| \nonumber \\
(i, j &= 1, 2, \ldots, n; i \neq j). 
\end{align}

(9)

Now we may assume that for each $i$, $|x_i| < 1$, since otherwise $\theta = a_{ii}$ for some $i$ and then clearly $\theta$ lies inside the ovals (6). Letting $|x_i|^q = a_i$ and $(k_i + 1)^{-1} = b_i$ for $i = 1, 2, \ldots, n$, we see by the lemma that there exist integers $u$ and $v$ for which

\begin{align}
|a_{ii} - a_{vv}| & \leq a_i b_i (1 - |x_i|^q)(1 - |x_v|^q) \\
& \leq (k_i + 1)^{-1}(k_v + 1)^{-1}(1 - |x_i|^q)(1 - |x_v|^q).
\end{align}

(10)

It follows that

\begin{align}
|a_{uu} - \theta| |a_{vv} - \theta| & \leq (k_u k_v)^{\frac{1}{q}} R(u, p) R(v, p), 
\end{align}

(11)

and the theorem is proved.

Consideration of the case $\theta = 0$ in the theorem leads to the following corollary:

**Corollary 1.** The $n \times n$ matrix $A$ is nonsingular provided that positive numbers $k_1, k_2, \ldots, k_n, p$ and $q$ satisfying (3) and (4) can be found such that

\begin{align}
|a_{ii}| |a_{jj}| & > (k_i k_j)^{\frac{1}{q}} R(i, p) R(j, p) \\
\end{align}

for each pair $i, j = 1, 2, \ldots, n; i \neq j$.

Consideration of the limiting case where $p$ approaches 1 and $q$ becomes infinite yields Brauer's theorem (2). If we reverse this procedure and let $p$ become infinite while $q$ approaches 1, we obtain the following corollary:

**Corollary 2.** Let $A = (a_{ij})$ and $k_1, k_2, \ldots, k_n$ be as above. Let
Then the characteristic roots of $A$ lie in the union of the ovals

$$|a_{ii} - z| |a_{jj} - z| \leq k_i k_j m_i m_j$$

$(i, j = 1, 2, \ldots, n; i \neq j)$.

**Proof.** Since $R(\varrho, \rho) \leq (m_i^p(n-1)^{1/p})^{m_i(n-1)^{1/p}}$, the result is immediate.

From Corollary 2 we obtain the following nonsingularity condition:

**Corollary 3.** The $n \times n$ matrix $A = (a_{ii})$ is nonsingular provided that positive numbers $k_1, k_2, \ldots, k_n$ satisfying (3) can be found such that

$$|a_{ii}| |a_{jj}| > k_i k_j m_i m_j$$

for each pair $i, j = 1, 2, \ldots, n; i \neq j$.

**References**


