ON BERGMAN'S KERNEL FUNCTION FOR SOME UNIFORMLY ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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1. We present here a generalization of the theory of Bergman's kernel function for uniformly elliptic partial differential equations of the divergence type

\[ \nabla u \equiv \frac{\partial}{\partial x_k} (a_{ik} \partial u/\partial x_i) = 0. \]

It is known that for regular open sets \( \Omega \) in \( \mathbb{R}^n \) the expression

\[ M_\Omega(u) = \int_{\Omega} a_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} dX \]

is a natural norm on the space of regular solutions of \( \nabla u = 0 \) vanishing at a point \( x_0 \in \Omega \). It is proved that for \( E \) compact in \( \Omega, x \in E \)

\[ |u(x)|^2 \leq K(E) M_\Omega(u). \]

The existence of Bergman's kernel \( K(x, y) \) and the convergence of its expansion in terms of a complete orthonormal set of functions follows at once. We prove the boundedness of \( K(x, y) \) on compact subsets of \( \Omega \). A sharp value for \( K(E) \) is found to be \( \sup_{B} K(x, x) \).

2. We consider partial differential equations of the type

\[ \nabla u = \frac{\partial}{\partial x_k} (a_{ik} \partial u/\partial x_i) = 0, \quad i, k = 1, \cdots, n, \]

where the coefficients \( a_{ik} \in C^{(1,1)} \) in a regular region \( \Omega \subset \mathbb{R}^n \). Moreover, \( \nabla \) satisfies a uniform ellipticity condition

\[ \lambda^{-1} \sum_{i=1}^{n} \xi_i^2 \leq a_{ik} \xi_k \xi_i \leq \lambda \sum_{i=1}^{n} \xi_i^2. \]

We recall the definition of regularity: let \( \Omega \) be a subregion of a region \( V \subset \mathbb{R}^n \). Let \( B \) be the open unit ball centered at the origin and let \( P \) be the hyperplane \( x_n = 0 \). \( \Omega \) shall be called a regular subregion [1] if:

(I) \( \text{Bd } \Omega \) is compact in \( V \),

(II) every \( x \in \text{Bd } \Omega \) has a neighborhood \( N(x) \) and a diffeomorphism \( h: N(x) \rightarrow B \) such that \( h(N(x) \cap \text{Bd } \Omega) = B \cap P \) and \( h(N(x) \cap \Omega) \) is one
of the two half balls of $B - P$,
(III) $\bar{\Omega}$ is compact in $V$,
(IV) $\Omega$ and $V - \bar{\Omega}$ have the same boundary in $V$,
(V) each component of $V - \Omega$ is noncompact in $V$.

We assume moreover that $h \in C^{1,\lambda}$.

We shall use the following lemmas as applied to regular solutions in $\Omega$ of $\mathcal{M}u = 0$.

**Lemma I (Poincaré) [2]**. If $w, w_1, \ldots, w_n$ are square integrable in a ball $B_R$ of radius $R$, and if $\bar{w}$ is the average of $w$ over $B_R$, then

$$\int_{B_R} (w - \bar{w})^2 \, dX \leq C(B_R) \sum_{i=1}^n (w_i)^2 \, dX,$$

where $C(B_R)$ denotes a constant which depends only on $B_R$.

**Lemma II (J. Moser) [3]**. If $u$ is a solution of $\mathcal{M}u = 0$ which is defined in $|x| < 2R$ then, for $|x| \leq R$

$$u^2(x) \leq CR^{-n} \int_{|x|<2R} u^2 \, dX,$$

where $C$ denotes a constant.

We now give a bound for the first derivatives of a regular solution of $\mathcal{M}u = 0$ in terms of

$$M_a(u) = \int_a a_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \, dX.$$

**Theorem I.** Let $E$ be a compact subset of $\Omega$. Then for $x \in E$, $u$ a regular solution of $\mathcal{M}u = 0$, one has

$$|\frac{\partial u}{\partial x_k}|_E \leq C(E)M_a^{1/2}(u),$$

where $C(E)$ denotes a constant depending only on $E$.

**Proof.** Let $\delta > 0$ be defined such that the distance from $E$ to $\text{Bd} \, \Omega$ is greater than $4\delta$. If we denote by $B(x; R)$ the ball of center $x$ and radius $R$,

$$B(x, 4\delta) \subset \Omega \quad \forall x \in E.$$

Let $x_0$ be a point of $E$, and let $G(x; y)$ be Green's function for $B(x_0, 2\delta)$. Then:

$$u(x) = \int_{\text{Bd} \, B(x_0, 2\delta)} u(y) \frac{\partial}{\partial n_y} G(y; x) \, d\sigma,$$
where $\partial/\partial v^*$ denotes the conormal derivative. Hence (cf. [4])

$$\frac{\partial u}{\partial x_k} = \int_{B(d, 2\delta)} u(y) \frac{\partial}{\partial x_k} \frac{\partial}{\partial v^*} G(y; x) dy,$$

$x \in B(x_0, 2\delta)$. Let

$$\tilde{u}(x_0; 2\delta) = \int_{B(x_0, 2\delta)} u(x) dx / \int_{B(x_0, 2\delta)} dx,$$

then

$$\frac{\partial u}{\partial x_k} = \int_{B(d, 2\delta)} (u(y) - \tilde{u}(x_0; 2\delta)) \frac{\partial}{\partial x_k} \frac{\partial}{\partial v^*} G(y; x) dy,$$

$x \in B(x_0, 2\delta)$.

$$|\frac{\partial u}{\partial x_k}| \leq C \max_{B(d, 2\delta)} |u(y) - \tilde{u}(x_0; 2\delta)| \int_{B(d, 2\delta)} \frac{dy}{|x - y|^{n-1}},$$

$x \in B(x_0, 2\delta)$.

Let $\omega_n$ be the area of the $n-1$ sphere:

$$|\frac{\partial u}{\partial x_k}| \leq C \omega_n^{-1} \max_{B(d, 2\delta)} |u(x) - \tilde{u}(x_0; 2\delta)|, \quad x \in B(x_0, \delta),$$

by the maximum principle. By Lemma II

$$|\frac{\partial x}{\partial x_k}|^2 \leq C \int_{B(d, 2\delta)} (u(x) - \tilde{u}(x_0, 2\delta))^2 dx \quad x \in B(x_0, \delta),$$

and by Lemma I

$$|\frac{\partial u}{\partial x_k}|^2 \leq C \int_{B(d, 2\delta)} \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 dx$$

$$\leq C \lambda \int_{B(d, 2\delta)} \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 dx$$

$$\leq C \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 dx$$

and $C$ depends only on $\omega_n$ and $\delta$. Cover now $E$ by a finite number, say $N$, of balls $B(x_j; \delta), j = 1, \ldots, n$.

Then

$$|\frac{\partial u}{\partial x_k}|^2 \leq C M_0(u), \quad x \in E,$$

where $C = \max_j C$. 

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3. Let \( \Omega \) be regular, and let \( x_0 \) be fixed in \( \Omega \). Consider a compact set \( E \subset \Omega \). Let \( 4\delta \) be a positive number smaller than the distance from \( E \cup \{ x_0 \} \) to \( \partial \Omega \). Cover \( E \cup \{ x_0 \} \) by a finite number of open balls of radius \( \delta \), \( B(x_0, \delta), \ldots, B(x_N, \delta) \). A point in each \( B(x_i; \delta) \) can be joined to \( x_0 \) by an arc \( \gamma_i \) in \( \Omega \). Let \( 4\delta' \) be a positive number smaller than the distance from \( \bigcup \{ B(x_i; \delta) \} \cup \gamma_i \) to \( \partial \Omega \) and cover each \( \gamma_i \) by a finite number of open balls of radius \( \delta' \), say \( B(y_i, \delta') \).

**Corollary.** Let \( u \) be a regular solution of \( \Re \mu = 0 \) vanishing at \( x = x_0 \), then for \( x \in E \)

\[
|u(x)|^2 \leq K(E)M_a(u)
\]

where \( K(E) \) depends only on \( E \) (and on \( x_0 \)).

**Proof.** It follows from the theorem that if \( B(\tilde{x}, 4\delta'') \subset \Omega \) where \( \text{dist}(\tilde{x}, \partial \Omega) > 4\delta'' \) then for \( x \) such that \( |x - \tilde{x}| < \delta \)

\[
|\grad u|^2 \leq C(\tilde{x}, \delta'')M_B(\tilde{x}, 4\delta')(u).
\]

Let \( x', x'' \) be points in \( B(\tilde{x}, \delta'') \) then

\[
|u(x') - u(x'')| \leq \int_{x'}^{x''} |\grad u| \, ds \leq 2\delta''C^{1/2}(\tilde{x}, \delta'')M_{B(\tilde{x}, 4\delta')}(u).
\]

Applying the last inequality to the covering defined by \( B(x_i, 4\delta) \) and \( B(y_j, 4\delta') \) one gets

\[
|u(x) - u(x_0)|^2 = |u(x)|^2 \leq K(E)M_a(u),
\]

which proves the corollary.

From the corollary and from the general theory [5] we get immediately the existence of a complete orthonormal system \( \{ \phi_r(x) \} \) and an expansion for Bergman's kernel

\[
K(x, y) = \sum_{r=1}^{\infty} \phi_r(x)\phi_r(y),
\]

which for fixed \( x \) converges uniformly on compact subsets of \( \Omega \). The \( \phi_r(x) \) may be chosen so that \( \phi_r(x_0) = 0 \) \( \forall r \).

As an application we shall prove the following theorem.

**4. Theorem II.** The function \( K(x, x) \) is bounded on every compact subset \( E \) of \( \Omega \).

**Proof.** Cf. [6]. From Theorem I we get, for fixed \( k \):

\[
|\partial u/\partial x_k|_E \leq C(E)M_{1/2}^a(u).
\]
If \( u \) is a solution of \( \Delta u = 0 \), regular and such that \( \partial u / \partial x_k = 1 \) at \( x_0 \in E \), then \( M_0(u) \leq 1/C^2(E) \).

Consider the function

\[
\phi^*(x) = \lambda^{-1/2} \sum_{k=1}^{N} \frac{\partial \phi^*/\partial x_k(x_0) \phi^*(x)}{\sum_{k=1}^{N} [\partial \phi^*/\partial x_k]^2},
\]

\( \Delta \phi^* = 0 \) and \( \partial \phi^*/\partial x_k(x_0) = 1 \).

Therefore

\[
M_0(\phi^*) \leq \lambda \int_0^1 |\text{grad } \phi^*|^2 dX = 1 / \sum_{k=1}^{N} [\partial \phi^*/\partial x_k]^2.
\]

Therefore

\[
\sum_{k=1}^{N} [\partial \phi^*/\partial x_k(x_0)]^2 \leq C^2(E),
\]

and

\[
\sum_{k=1}^{N} [\partial \phi^*/\partial x_k(x_0)]^2 \leq C^2(E),
\]

and this is true for all \( x_0 \in E \). An analogous proof works for all \( k \), \( k = 1, \ldots, n \).

Now, we have

\[
\sum_{k=1}^{N} [\phi_k(x)]^2 = \sum_{k=1}^{N} [\phi_k(x) - \phi_k(x_0)]^2
\]

and

\[
[\phi_k(x) - \phi_k(x_0)]^2 \leq \left[ \int_{\gamma(x_0, x)} |\text{grad } \phi_k|^2 ds \right]^2
\]

\[
\leq 4L^2 \int_{\gamma(x_0, x)} |\text{grad } \phi_k|^2 ds
\]

where \( \gamma(x_0, x) \) is an arc from \( x_0 \) to \( x \), lying in \( \Omega \) and of length \( L \); therefore

\[
\sum_{k=1}^{N} [\phi_k(x)]^2 \leq 4L^2C^2(E),
\]

and
\[ \sum_{r=1}^{\infty} [\phi_r(x)]^2 = K(x, x) \leq 4L^2C^2(E). \]

We are now ready to give the best estimate for \( K(E) \) in the corollary. Let \( p_0 \) and \( p_1 \) be the principal functions for \( \Re u = 0 \) and \( \Omega \), defined as in [1]. From Theorem 6 in [1],

\[ |u(x)|^2 \leq M_0(p_0 - p_1)M_0(u) \]

with equality only for \( u = a(p_0 - p_1), a \in \mathbb{R} \). Moreover, Theorem 5 in [1] shows that

\[ M_0(u) - 2u(x) = M_0(p_0 - p_1) + M(u - p_0 + p_1), \]

or

\[ u(x) = M_0(u, p_0 - p_1). \]

It follows that \( p_0 - p_1 \), which vanishes at \( x = x_0 \) is the Bergman kernel for the space of regular solutions of \( \Re u = 0 \) in \( \Omega \) vanishing at \( x_0 \),

\[ |u(x)|^2 \leq K(x, x)M_0(u), \]

and \( \sup K(x, x) \) is the best possible value for \( K(E) \).

Another application of the previous results would be the obtention of the extremal properties of principal functions [1] for open regions \( V \) of \( \mathbb{R}^n \), such that there exists a nested sequence of regular \( \{\Omega_n\} \), with the properties \( \Omega_{n+1} \supseteq \Omega_n \) and \( \bigcup \Omega_n = V \).

**Bibliography**


