HOMEOMORPHIC CONJUGACY OF AUTOMORPHISMS
ON THE TORUS

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Introduction. Let \( \gamma \) be a continuous map of the \( n \)-dimensional torus \( T^n = \mathbb{R}^n / \mathbb{Z}^n \) into itself where \( \mathbb{R}^n \) is an \( n \)-dimensional real Euclidean space and \( \mathbb{Z}^n \) is the subgroup of \( \mathbb{R}^n \) with integral coordinates. Let \( \pi: R^n \to T^n \) denote the universal covering map. There is a unique \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) with \( 0 \leq c_i < 1, \ i = 1, \ldots, n \), such that \( \pi(c) = \gamma(0) \) and a unique continuous map \( F: R^n \to R^n \) with \( F(0) = c \) which is a “lifting” of \( \gamma \), i.e., which satisfies \( \pi F = \gamma \pi \). If we put \( G(x) = F(x) - c, \ x \in \mathbb{R}^n \), then \( G|\mathbb{Z}^n \) is a homomorphism of \( \mathbb{Z}^n \) into itself and therefore extends uniquely to a linear map \( L: R^n \to R^n \). In fact making the canonical identification of \( \mathbb{Z}^n \) with the fundamental group \( \pi_1(T^n) \) of \( T^n \), \( G|\mathbb{Z}^n \) is just the homomorphism of \( \pi_1(T^n) \) induced by \( \gamma \). It follows that if \( \gamma \) is a homeomorphism then \( G|\mathbb{Z}^n \) is an automorphism of \( \mathbb{Z}^n \) hence \( L \in \text{SL}(n, \mathbb{Z}) \), the group of linear automorphisms of \( \mathbb{R}^n \) whose matrices are unimodular, i.e., have determinant \( \pm 1 \) and integer entries.

We next note that

\[
F(x) = L(x) + P(x) + c
\]

where \( P: R^n \to R^n \) is a continuous periodic map (i.e., \( P(x+v) = P(x) \), \( x \in \mathbb{R}^n, v \in \mathbb{Z}^n \)) satisfying \( P(0) = 0 \). This fact is established by considering \( P(x) = F(x) - L(x) - c \). Clearly \( P(0) = 0 \). In view of the fact that \( L\nu \in \mathbb{Z}^n \) whenever \( \nu \in \mathbb{Z}^n \) we have \( \pi(P(x+v) - P(x)) = \pi(F(x+v) - F(x) - L\nu) = \gamma(\pi x + \pi v) - \gamma(\pi x) - \pi L\nu = \gamma(\pi x) - \gamma(\pi x) = 0 \); consequently \( P(x+v) - P(x) \) is always in \( \mathbb{Z}^n \). Since \( \mathbb{R}^n \) is connected and \( \mathbb{Z}^n \) is discrete \( P(x+v) - P(x) \) is (for \( v \) fixed) independent of \( x \). Thus \( P(x+v) - P(x) = P(0) + v = P(0) = P(0) = F(v) - L\nu - c = G(v) - L\nu \) which is zero by definition \( L|\mathbb{Z}^n = G|\mathbb{Z}^n \). We shall call \( L \) the linear part, \( P \) the periodic part, and \( c \) the constant part of the lifting \( F \) and when necessary place subscripts on these symbols to indicate the mapping on \( T^n \) from which they came.

The linear, periodic, and constant part of a lifting are unique; for let \( L', P', c' \), be other ones. \( F(x) = L(x) + P(x) + c = L'(x) + P'(x) + c' \), yields \( L(x) - L'(x) = P(x) - P'(x) + c - c' \). The right-hand side of the

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\footnote{1} It has come to the attention of the authors that the following overlaps the present work: D. Z. Arov, Topological similitude of automorphisms and translations of compact commutative groups, Uspekhi Mat. Nauk 18 (1963), no. 5 (113), 135–138. (Russian)
last relation is linear while the left is periodic. This can only occur if 
$L-L'=0$. Then since $P(0)=P'(0)=0$, it follows that $c=c'$ and 
$P=P'$.

If $\gamma$ is a continuous automorphism of $T^n$, then $F_\gamma$ is linear; thus its 
periodic and constant parts vanish so that $F_\gamma=L_\gamma$. Also every 
$L\in\text{SL}(n, Z)$ is the lifting of a continuous automorphism on $T^n$. The 
correspondence $\gamma\rightarrow L_\gamma$ is an isomorphism of the group $\text{Aut}(T^n)$ with 
$\text{SL}(n, Z)$. More generally the mapping $\gamma\rightarrow L_\gamma$ is a homomorphism 
of the group $\text{Homeo}(T^n)$ onto $\text{SL}(n, Z)$. We shall prove this by 
showing that $L_{\alpha\beta}=L_\alpha L_\beta$ for any two homeomorphisms $\alpha$ and $\beta$ of $T^n$ 
on to itself. By uniqueness of lifting

$$(2)\quad F_{\alpha\beta} = F_\alpha F_\beta.$$

On one hand,

$$(3)\quad F_{\alpha\beta}(x) = L_{\alpha\beta}(x) + P_{\alpha\beta}(x) + c_{\alpha\beta};$$

on the other hand, using (1) and adding and subtracting $P_\alpha(c_\beta)$,

$$F_\alpha F_\beta(x) = L_\alpha L_\beta(x) + [L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta)]
+ L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha.$$

Because $L_\beta Z^n \subseteq Z^n$ the term in the brackets is periodic. This term 
vanishes when $x=0$ so that it is a periodic part of the lifting $F_{\alpha\beta}$. 
From the uniqueness of the various parts of a lifting

$$(5)\quad L_{\alpha\beta} = L_\alpha L_\beta,$$

$$(6)\quad P_{\alpha\beta}(x) = L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta),$$

$$(7)\quad c_{\alpha\beta} = L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha.$$

Finally if $\gamma$ is a continuous automorphism of $T^n$, it preserves Haar 
measure on $T^n$ and to such transformations we can apply the notions 
of ergodic theory [1].

**THEOREM.** If $\alpha$ and $\beta$ are continuous automorphisms of $T^n$ such that

$$(8)\quad \gamma \alpha \gamma^{-1} = \beta$$

where $\gamma$ is a homeomorphism of $T^n$ onto itself then

(i) $L_\gamma L_\alpha L_\gamma^{-1} = L_\beta$ (\(\alpha\) and $\beta$ are conjugate elements in the group of 
measure preserving transformations on $T^n$).

(ii) $c_\gamma$ is a fixed point of $L_\beta$ (\(\gamma(0)\) is a fixed point of $\beta$).

(iii) If $\alpha$ is ergodic then $P_\gamma = 0$ (\(\gamma\) is a continuous automorphism of 
$T^n$ composed with a rotation. The rotation is by a fixed point of $\beta$ and the 
continuous automorphism satisfies the conjugacy relation between 
$\alpha$ and $\beta$).
PROOF. Relation (i) follows immediately from (5).
From (7)
\[ c_{\gamma a} = L_{\gamma}(c_a) + P_{\gamma}(c_a) + c_{\gamma} \]
and
\[ c_{\beta \gamma} = L_{\beta}(c_{\gamma}) + P_{\beta}(c_{\gamma}) + c_{\beta}. \]

From (8) \( \gamma a = \beta \gamma \). Since the constant part of the lifting for this mapping is unique, \( c_{\gamma a} = c_{\beta \gamma} \).

Therefore because \( P_{\beta} = 0 \) and \( c_a = c_{\beta} = 0 \), we have statement (ii) that \( L_{\beta}(c_{\gamma}) = c_{\gamma} \).

It is convenient to prove (iii) in the following steps.

Step I. Define the function \( Q \) by \( Q(x) = L_{\gamma}^{-1}P_{\gamma}(x) \). Then \( QL_{\gamma}^m = L_{\gamma}^mQ \) for all \( m \in \mathbb{Z} \). To prove this we first derive from (6) that \( P_{\gamma a}(x) = P_{\gamma L_{\gamma}(x)} \) and \( P_{\beta \gamma}(x) = L_{\beta}P_{\gamma}(x) \). By hypothesis (8) and uniqueness of periodic part \( P_{\gamma L_{\gamma}} = L_{\beta}P_{\gamma} \). Substituting (i) in this expression \( P_{\gamma L_{\gamma}} = L_{\beta}L_{\gamma}^{-1}P_{\gamma} \), so that \( QL_{\gamma} = L_{\gamma}Q \). Thereupon Step I is obtained by induction on \( m \).

Step II. We next recall that if \( L: \mathbb{R}^n \to \mathbb{R}^n \) is an invertible linear operator and if \( \{ L^m x : m \in \mathbb{Z} \} \) is bounded then either \( x = 0 \) or \( \{ L^m x : m \in \mathbb{Z} \} \) is bounded away from zero. One way to see this is to express \( x \) in a basis that puts \( L \) in Jordan form. It can be verified that the hypothesis \( \{ L^m x : m \in \mathbb{Z} \} \) is bounded implies that \( x \) is a linear combination of characteristic vectors of \( L \) belonging to characteristic values of absolute value one.

Step III. If \( \{ L_{\gamma}^m Q(x) : m \in \mathbb{Z} \} \) is not bounded away from zero then \( Q(x) = 0 \). This follows from Step II. The set \( \{ L_{\gamma}^m Q(x) = QL_{\gamma}^m(x) : m \in \mathbb{Z} \} \) is bounded because \( Q \) being continuous and periodic is bounded.

Step IV. If \( \{ L_{\gamma a}^m x : m \in \mathbb{Z} \} \) is not bounded away from \( \mathbb{R}^n \) then \( Q(x) = 0 \). Since \( Q \) is periodic, \( L_{\gamma a}^m Q(x) = L_{\gamma a}^m(x) = Q(L_{\gamma a}^m(x) - \nu) \) for \( \nu \in \mathbb{R}^n \). By hypothesis there exists a subsequence \( \{ \nu_i : i = 1, 2, \cdots \} \) of \( \mathbb{Z} \) and a subset \( \{ \nu_i : i = 1, 2, \cdots \} \) of \( Z^s \) such that \( L_{\gamma a}^m x - \nu_i \to 0 \), \( i \to \infty \). Since \( Q \) is continuous and \( Q(0) = 0 \), \( L_{\gamma a}^m Q(x) = Q(L_{\gamma a}^m(x) - \nu_i) \to 0 \). By Step III \( Q(x) = 0 \).

Now \( \alpha \) is ergodic and so almost all orbits of \( \alpha \) are dense in \( T^s \). In particular the zero element in \( T^s \) is a limit point for almost all orbits or in other words \( \{ L_{\gamma a}^m x : m \in \mathbb{Z} \} \) is not bounded away from \( \mathbb{R}^n \) for almost all \( x \in \mathbb{R}^n \). From Step IV \( Q(x) = 0 \) almost everywhere. By continuity \( Q = 0 \) and since \( L_{\gamma}^{-1} \) is nonsingular \( P_{\gamma} = 0 \).

Remarks. Questions arise whether the theorem holds under weaker hypotheses. In (iii) one cannot merely drop the assumption of ergodicity, for choosing \( \alpha \) to be the identity transformation removes all
restrictions on the homeomorphism $\gamma$. Another question is whether the theorem holds if $\gamma$ is a measure preserving transformation instead of a homeomorphism. A positive answer would be significant in ergodic theory, for then an example could be constructed of two Kolmogoroff transformations [2] with the same entropy but which are not conjugate.

REFERENCES