References


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EXTENSION OF NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS

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Let $X$ be a Stein manifold, and let $A$ be an analytic subset of $X$. A well-known application of Cartan's Theorem B [2, Théorème 3 p. 52] states that each holomorphic function on $A$ is the restriction of a holomorphic function on $X$. This paper presents a generalization of this application, namely that each normal family of holomorphic functions on $A$ is the restriction of a normal family of holomorphic functions on $X$.

1. Let $X$ be a topological space which is $\sigma$-compact, i.e., the union of a countable family of compact sets. Let $K(X)$ denote the set of all compact subsets of $X$. For $K \in K(X)$ and $f: X \to \mathbb{C}$ define $\|f\|_K = \sup \{|f(x)| : x \in K\}$. Define

$$B(X) = \{f \mid f: X \to \mathbb{C}, \|f\|_K < \infty \text{ for all } K \in K(X)\}.$$

Clearly $B(X)$ is a complex vector space, and $\{\|\|_K \mid K \in K(X)\}$ is a family of pseudonorms on $B(X)$ which then becomes a locally convex vector space. Since $X$ is $\sigma$-compact, $B(X)$ is metrizable, and it is readily checked to be a Fréchet space.

Definition. Let $V$ be a vector subspace of $B(X)$. We say that a set $F \subset V$ is normal with respect to $V$ iff every sequence in $F$ has a subsequence which converges in $V$. 

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Lemma 1. Let $V$ be a vector subspace of $B(X)$. Then $F \subset V$ is normal with respect to $V$ if and only if $F$ is a compact subset of $V$.

We say that $F \subset B(X)$ is uniformly bounded on compact sets iff $\sup\{\|f\|_K \mid f \in F\} < \infty$ for all $K \subset K(X)$. If $V$ is a vector subspace of $B(X)$ and $F \subset V$ is normal with respect to $V$, then $F$ is uniformly bounded on compact sets.

Definition. Let $V$ be a vector subspace of $B(X)$. We say that the Vitali theorem holds for $V$ iff every set $F \subset V$ which is uniformly bounded on compact sets is normal with respect to $V$.

Lemma 2. Let $V$ be a vector subspace of $B(X)$. If the Vitali theorem holds for $V$, then $V$ is a closed subset of $B(X)$, i.e., $V$ is a Fréchet space with the induced pseudonorms.

Proof. Let $f \in V$. Then there exists a sequence $\{f_n\}$ in $V$ such that $f_n \to f$ for $n \to \infty$. Take $K \subset K(X)$. There exists an integer $n_0$ such that $\|f_n - f\|_K < 1$ for $n \geq n_0$. Let $M = \max\{\|f_n\|_K + 1 \mid n = 1, \cdots, n_0\}$. Then $\|f_n\|_K \leq M$ for $n = 1, 2, \cdots$. Hence $\sup\{\|f_n\|_K \mid n = 1, 2, \cdots\} \leq M < \infty$, that is $\{f_n\}_{n=1, 2, \cdots}$ is uniformly bounded on compact sets. Since the Vitali theorem holds for $V$, $\{f_n\}$ has a subsequence which converges in $V$. The limit of this subsequence must be $f$. Hence $f \in V$.

Lemma 3. Let $A$ be a closed subset of the $\sigma$-compact space $X$. Define $R:B(X) \to B(A)$ by $R(f) = f|A$ for $f \in B(X)$. Then $R$ is continuous and linear.

Proof. Clearly $R$ is linear. Each compact subset of $A$ is also compact in $X$. Hence each pseudonorm in $B(A)$ is the restriction of a pseudonorm in $B(X)$. Therefore $R$ is continuous.

Lemma 4. Let $E$ and $F$ be Fréchet spaces, and let $u:E \to F$ be continuous, linear, and surjective. Let $K$ be a compact subset of $F$. Then there exists a compact subset $K'$ of $E$ such that $u(K') = K$.

Proof. Let $G = u^{-1}(0)$. Then $G$ is a closed linear subspace of $E$, hence $E/G$ is a Fréchet space. Let $r:E \to E/G$ be the residual map, and define $v:E/G \to F$ by $v \circ r = u$. Then $v$ is continuous, linear, and bijective. According to [4, Satz (1), p. 170], $v$ is a topological isomorphism. The lemma then follows from [4, Satz (7), p. 281].

2. If $X$ is a complex space, we define $H(X)$ to be the set of holomorphic functions on $X$. As has been proved by Gunning [3] and Andreotti and Stoll [1, pp. 326–327], the Vitali theorem holds for
$H(X)$. We say that $F \subseteq H(X)$ is a normal family of holomorphic functions iff $F$ is normal with respect to $H(X)$.

**Theorem.** Let $X$ be a Stein manifold, and let $A$ be an analytic subset of $X$. Let $F = \{f_\lambda | \lambda \in \Lambda\}$ be a normal family of holomorphic functions on $A$. Then there exists a normal family $G = \{g_\lambda | \lambda \in \Lambda\}$ of holomorphic functions on $X$ such that $g_\lambda | A = f_\lambda$ for $\lambda \in \Lambda$.

**Proof.** According to the well-known application of Theorem B of Cartan referred to at the beginning of this paper, the restriction map $R: H(X) \to H(A)$, defined by $R(f) = f | A$, is onto. By Lemma 3, $R$ is continuous and linear. Since the Vitali theorem holds for $H(X)$ and $H(A)$, they are Fréchet spaces (Lemma 2). Application of Lemma 1 and Lemma 4 completes the proof.

**References**


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