TWO ELEMENTARY THEOREMS ON THE INTERPOLATION OF LINEAR OPERATORS

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Both theorems have to do with functions satisfying Hölder conditions.

Definition. Let $T$ be an operator which takes functions whose domain is $n$-space into functions whose domain is a metric space. $T$ will be said to be of Hölder type $(\alpha, \beta)$ norm $N$ if for $g = Tf$,

\[ |f(x) - f(x - h)| \leq A |h|^\alpha \quad \text{for all } x \text{ and } h, \]

implies

\[ |g(u) - g(v)| \leq NA |u - v|^\beta \quad \text{for all } u \text{ and } v. \]

(Throughout this paper, when dealing with a metric space we shall denote the distance between $u$ and $v$ by $|u - v|$.)

Theorem 1. Suppose that $0 \leq \alpha_0 \leq \alpha_1 \leq 1$, $\beta_0 \geq 0$, $\beta_1 \geq 0$ and that $T$ is a linear operator taking functions whose domain is $n$-space into functions whose domain is a metric space. If $T$ is simultaneously of Hölder type $(\alpha_0, \beta_0)$ norm $N_0$ and of Hölder type $(\alpha_1, \beta_1)$ norm $N_1$ and if $0 \leq t \leq 1$, then $T$ is of Hölder type $(\alpha, \beta)$ norm $N$ where

\[ \alpha = \alpha_t = \alpha_0(1 - t) + \alpha_1t, \]
\[ \beta = \beta_t = \beta_0(1 - t) + \beta_1t, \]
\[ N \leq R_n N_0^{1-t} N_1^t, \]

and where $R_n$ depends only on the dimension of $n$-space.

Proof. Without loss of generality we may assume

\[ |f(x) - f(x - h)| \leq |h|^\alpha. \]

We first prove the theorem in case the domain of $f$ is the real line, that is when $n = 1$.

For $r > 0$, let

\[ K_r(s) = \begin{cases} \frac{1}{r} - \frac{|s|}{r^2} & \text{if } |s| < r, \\ 0 & \text{if } |s| \geq r. \end{cases} \]

Then

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\[ \int K_r(s) \, ds = \int_{|s| < r} K_r(s) \, ds = 1, \]
\[ \int K'_r(s) \, ds = 0, \]
and,
\[ \int \left| K'_r(s) \right| \, ds = \frac{2}{r}. \]

Let
\[ f_r(x) = \int f(x - s)K_r(s) \, ds = \int f(s)K_r(x - s) \, ds. \]

Let \( e_r(x) = f(x) - f_r(x), \ g = Tf, \ g_r = Tf_r, \) and \( \eta_r = Te_r. \) Then \( g = g_r + \eta_r \)
by linearity of \( T. \)

\[ f'_r(x) = \int f(s)K'_r(x - s) \, ds = \int f(x - s)K'_r(s) \, ds = \int (f(x - s) - f(x))K'_r(s) \, ds. \]

\[ |f'_r(x)| \leq \int_{|s| < r} |f(x - s) - f(x)| |K'_r(s)| \, ds \leq r^\alpha (2/r) = 2r^{\alpha - 1}. \]

Case 1. \( |h| < r. \)

\[ |f_r(x) - f_r(x - h)| = \leq |h| \sup_{y} |f'_r(y)| \leq 2|h| r^{\alpha - 1} \]
\[ = 2|h|^{\alpha_1} |h|^{1 - \alpha_1 \alpha - 1} \leq 2|h|^{\alpha_1 \alpha - \alpha}. \]

Case 2. \( |h| \geq r. \)

\[ |f_r(x) - f_r(x - h)| = \left| \int (f(x - s) - f(x - h - s))K_r(s) \, ds \right| \]
\[ \leq |h|^\alpha = |h|^{\alpha_1} |h|^{\alpha - \alpha_1} \leq |h|^{\alpha_1 \alpha - \alpha}. \]

In either case, \( f_r \) satisfies a H"older condition of order \( \alpha_1, \) indeed,
\[ |f_r(x) - f_r(x - h)| \leq 2r^{\alpha - \alpha_1} |h|^{\alpha_1}. \]

Thus,
\[ |g_r(u) - g_r(v)| \leq N_1 2r^{\alpha - \alpha_1} |u - v|^{\beta_1}. \]
\[ \epsilon_r(x) = f(x) - f_r(x) = \int (f(x) - f(x - s))K_r(s) \, ds. \]

\[ |\epsilon_r(x)| \leq \int_{|s|<r} |s|^\alpha K_r(s) \, ds \leq r^\alpha. \]

**Case 1.** \(|h| \geq r.\)

\[ |\epsilon_r(x) - \epsilon_r(x - h)| \leq 2r^\alpha \leq 2r^{\alpha - \alpha_0} |h|^\alpha_0. \]

**Case 2.** \(|h| < r.\)

\[ |\epsilon_r(x) - \epsilon_r(x - h)| \leq |f(x) - f(x - h)| + |f_r(x) - f_r(x - h)| \leq |h|^\alpha + |h|^\alpha \leq 2|h|^\alpha_0 r^{\alpha - \alpha_0}. \]

Thus \(\epsilon_r\) satisfies a Hölder condition of order \(\alpha_0.\) Therefore,

\[ |\eta_r(u) - \eta_r(v)| \leq N_0 2r^{\alpha - \alpha_0} |u - v|^\beta_0. \]

Thus, if we set \(r = (N_1 |u - v|^{\beta_1 - \beta_0}/N_0)^{1/(\alpha_1 - \alpha_0)},\)

\[ |g(u) - g(v)| \leq |g_r(u) - g_r(v)| + |\eta_r(u) - \eta_r(v)| \leq 2N_1^{\alpha - \alpha_0} |u - v|^\beta_1 + 2N_0^{\alpha - \alpha_0} |u - v|^\beta_0 = 4N_0^{1-1}N_1^{1} |u - v|^\beta. \]

This proves the theorem when the domain of \(f\) is one dimensional. For \(n > 1,\) the case \(n = 2\) is already sufficiently general to illustrate the proof. In this case we let

\[ K_r(s) = \begin{cases} 3/\pi r^2 - 3 |s|/\pi r^3 & \text{if } |s| < r, \\ 0 & \text{if } |s| \geq r. \end{cases} \]

For a given \(h = (h_1, h_2) \neq 0,\) let \(\partial/\partial \theta\) denote directional differentiation in the direction \(\theta = h/|h|\). Then \([\partial/\partial \theta]K_r(s) ds\) vanishes and \([\partial/\partial \theta]K_r(s) ds = O(1/r).\) Thus,

\[ |(\partial/\partial \theta) f_r(x)| \leq \int_{|s|<r} |f(x - s) - f(x)| |(\partial/\partial \theta) K_r(s)| \, ds \leq r^\alpha O(1/r) = O(r^{\alpha - 1}). \]

Therefore, if \(0 < |h| < r, \theta = h/|h|,\)

\[ |f_r(x) - f_r(x - h)| \leq |h| \sup_y |(\partial/\partial \theta) f_r(y)| = |h| O(r^{\alpha - 1}) = O(|h|^{\alpha_0 r^{\alpha - \alpha_1}}). \]

The rest of the proof goes through as before.
Definition. An operator $T$ is said to take $L^p$ into $\text{Lip } \alpha$ with norm $N$ if for $g = Tf$,

$$\left| g(u) - g(v) \right| \leq N \|f\|_p \ |u - v|^\alpha \text{ for all } u \text{ and } v.$$

If $f$ is a measurable function and $\gamma > 0$, let

$$m(f, \gamma) = m(\|f\|, \gamma) = \text{measure of } \{x: |f(x)| > \gamma\}.$$

It is easily shown that

$$\int |f(x)| \, dx = \int_0^\infty m(f, \gamma) \, d\gamma.$$

Furthermore, for $p > 0$,

$$m(\|f\|_p, \gamma) = \text{meas}\{x: |f(x)|^p > \gamma\}$$

$$= \text{meas}\{x: |f(x)| > \gamma^{1/p}\} = m(f, \gamma^{1/p}).$$

Thus,

$$\|f\|_p^p = \int |f(x)|^p \, dx = \int_0^\infty m(\|f\|_p, \gamma) \, d\gamma$$

$$= \int_0^\infty m(f, \gamma^{1/p}) \, d\gamma = p \int_0^\infty m(f, \gamma)^{p-1} \, d\gamma.$$

Given $k \geq 0$, let

$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq k, \\ k \text{ sgn } f(x) & \text{if } |f(x)| > k, \end{cases}$$

and let

$$f(x) = f(x) - f_k(x).$$

Theorem 2. Suppose that $0 < p_0 \leq p_1 \leq \infty$, $\alpha_0 \geq 0$, $\alpha_1 \geq 0$, and that $T$ is a linear operator taking measurable functions on a measure space into functions whose domain is a metric space. If $T$ simultaneously takes $L^{p_0}$ into $\text{Lip } \alpha_0$ with norm $N_0$ and $L^{p_1}$ into $\text{Lip } \alpha_1$ with norm $N_1$ and if $0 \leq t \leq 1$, then $T$ takes $L^p$ into $\text{Lip } \alpha$ with norm $N$ where

$$1/p = 1/p_t = (1 - t)/p_0 + t/p_1,$$

$$\alpha = \alpha_t = (1 - t)\alpha_0 + t\alpha_1,$$

$$N \leq N_0^{1-t} N_1^t / (1 - t)^{1-t} t^t \leq 2N_0^{1-t} N_1^t.$$  

(It is to be remarked that $1/(1-t)^{1-t} t^t$ tends to 1 as $t$ tends to 0 or 1.)
Proof. Suppose, without loss of generality, that $\|f\|_p = 1$. Fix $k \geq 0$, then $f = f^* + f_k$. Let $g = Tf$, $g_0 = Tf^k$ and $g_1 = Tf_k$, then $g = g_0 + g_1$ by linearity of $T$.

\[
(\|f^k\|_{p_0})^{p_0} = p_0 \int_0^\infty y^{p_0 - 1}m(f^k, y) \, dy = p_0 \int_0^\infty y^{p_0 - 1}m(f, y + k) \, dy
\]
\[
= p_0 \int_k^\infty (z - k)^{p_0 - 1}m(f, z) \, dz \leq p_0 \int_k^\infty z^{p_0 - 1}m(f, z) \, dz
\]
\[
\leq p_0 k^{p_0 - p} \int_k^\infty z^{p_0 - 1}m(f, z) \, dz \leq (p_0 k^{p_0 - p}/p)(\|f\|_p)^p
\]
\[
= p_0 k^{p_0 - p}/p.
\]

Thus $\|f^k\|_p \leq (p_0/p)^{1/p_0} k^{1-p/p_0}$; since $T$ takes $L^{p_0}$ into $\text{Lip} \alpha_0$ with norm $N_0$,

\[
|g_0(u) - g_0(v)| \leq N_0(p_0/p)^{1/p_0} k^{1-p/p_0} |u - v|^\alpha_0.
\]

\[
(\|f^k\|_{p_1})^{p_1} = p_1 \int_0^\infty y^{p_1 - 1}m(f^k, y) \, dy = p_1 \int_0^k y^{p_1 - 1}m(f, y) \, dy
\]
\[
\leq p_1 k^{p_1 - p} \int_0^k y^{p_1 - 1}m(f, y) \, dy \leq p_1 k^{p_1 - p}/p.
\]

Thus $\|f^k\|_{p_1} \leq (p_1/p)^{1/p_1} k^{1-p/p_1}$, and this last equation is valid even if $p_1 = \infty$.

\[
|g_1(u) - g_1(v)| \leq N_1(p_1/p)^{1/p_1} k^{1-p/p_1} |u - v|^\alpha_1.
\]

If we set $A = 1/p_0 - 1/p_1$, then $1/p - 1/p_1 = A(1 - t)$ and $1/p_0 - 1/p = At$.

Thus, if we let

\[
k^{pA} = (t(1 - t)(p_0/p)^{1/p_0}(p_1/p)^{-1/p_1}(N_0/N_1) |u - v|^\alpha_0 - \alpha_1
\]

\[
|g(u) - g(v)| \leq |g_0(u) - g_0(v)| + |g_1(u) - g_1(v)|
\]
\[
= N_0(p_1/p)^{1/p_0} k^{pA} |u - v|^\alpha_0
\]
\[
+ N_1(p_1/p)^{1/p_1} k^{pA}(1-t) |u - v|^\alpha_1
\]
\[
= N_0^{1-t} N_1 (1/t(1-t)^{-1}) (p_0/p)^{1-t/p_0} (p_1/p)^{t/p_1} |u - v|^\alpha.
\]

Let

\[
B = (p_0/p)^{(1-t)/p_0} (p_1/p)^{t/p_1},
\]

then

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\[
\log B = \left(\frac{1}{p}\right) \log\left(\frac{1}{p}\right) - (1 - t)\left(\frac{1}{p_0}\right) \log\left(\frac{1}{p_0}\right) - t\left(\frac{1}{p_1}\right) \log\left(\frac{1}{p_1}\right).
\]

But \(x \log x\) is a convex function of \(x \geq 0\), so that
\[
(1/p) \log(1/p) \leq (1 - t)(1/p_0) \log(1/p_0) + t(1/p_1) \log(1/p_1).
\]

Thus \(\log B \leq 0\), \(B \leq 1\) and the theorem is established.

**Remark.** It is possible to strengthen the result of Theorem 2. We shall say that a measurable function \(f\) belongs to weak \(L^p\) if there exists a number \(A\) such that for all \(y > 0\),
\[
m(f, y) \leq (A/y)^p.
\]

If \(f \in L^p\) then \(f\) belongs to weak \(L^p\), since
\[
(||f||_p)^p = p \int_0^\infty m(f, u)u^{p-1} du \geq p \int_0^u m(f, u)u^{p-1} du
\]
\[
\geq pm(f, y) \int_0^u u^{p-1} du = m(f, y)y^p.
\]

Thus,
\[
m(f, y) \leq (||f||_p/y)^p.
\]

We shall say that a function \(f \in \text{Lip } \alpha\) if for all \(u\) and \(v\),
\[
|f(u) - f(v)| \leq A |u - v|^\alpha.
\]

We shall say \(f \in \text{Lip } \alpha\) if
\[
|f(u) - f(v)| = o(|u - v|^\alpha)
\]
as \(|u - v|\) tends to zero or infinity.

1°. Under the hypotheses of Theorem 2, \(T\) takes weak \(L^p\) into \(\text{Lip } \alpha\) if \(p_0 < p < p_1\).

2°. Under the hypotheses of Theorem 2, \(T\) takes \(L^p\) into \(\text{Lip } \alpha\) if \(p_0 < p < p_1\).

To prove 1°, we suppose that \(m(f, y) \leq 1/y^p\). Then
\[
(||f||_p)^p \leq p_0 \int_k^\infty z^{p_0-1} m(f, z) dz \leq (p_0/p - p_0)k^{p_0 - p},
\]
\[
||f_k||_{p_0} \leq (p_0/p - p_0)^{1/p_0}k^{1-p/p_0}.
\]

Similarly,
\[
||f_k||_{p_1} \leq (p_1/p_1 - p)^{1/p_1}k^{1-p/p_1}.
\]

Thus, if we let
\[ k^{pA} = \left( \frac{N_0}{N_1} \right) |u - v|^{\alpha_0 - \alpha_1}, \]

\[ |g(u) - g(v)| \leq N_0^{1-t} N_1^{1-t} |u - v|^{\alpha} \left\{ (p_0/p - p_0)^{1/p_0} + (p_1/p_1 - p)^{1/p_1} \right\}. \]

To prove 2°, we observe that

\[ \langle \|f\|_p \rangle^p = \int_0^\infty m(f, v^{1/p}) \, dv. \]

Since \( m(f, v^{1/p}) \) is a monotone function of \( v \), the finiteness of the integral implies

\[ m(f, v^{1/p}) = o(1/v) \text{ as } v \text{ tends to zero or infinity}. \]

Thus,

\[ m(f, y) = o(1/y^p) \text{ as } y \text{ tends to zero or infinity}. \]

Therefore,

\[ \|f_k\|_{p_0} = o(k^{1-p/p_0}) \text{ as } k \text{ tends to zero or infinity}, \]

and

\[ \|f_k\|_{p_1} = o(k^{1-p/p_1}) \text{ as } k \text{ tends to zero or infinity}. \]

Again we may let

\[ k^{pA} = \left( \frac{N_0}{N_1} \right) |u - v|^{\alpha_0 - \alpha_1}. \]

Thus,

\[ |g(u) - g(v)| = o(|u - v|^{\alpha}). \]

References

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