NOTE ON ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS

LOUIS J. RATLIFF, JR.1

All rings in this paper are assumed to be commutative rings with a unit element. If \( B \) is an ideal in a ring \( R \), the integral closure \( B_\alpha \) of \( B \) is the set of elements \( x \) in \( R \) such that \( x \) satisfies an equation of the form \( x^n + b_1 x^{n-1} + \cdots + b_n = 0 \), where \( b_i \in B \) (\( i = 1, \cdots, n \)). An ideal \( B \) in \( R \) is semi-prime in case \( B \) is an intersection of prime ideals. If \( R \) is an integral domain, then \( R \) is normal in case \( R \) is integrally closed in its quotient field. If \( P \) is a semi-local (Noetherian) ring, then \( P \) is analytically unramified in case the completion of \( P \) (with respect to the powers of the Jacobson radical of \( R \)) contains no nonzero nilpotent elements.

Let \( R \) be a semi-local ring with Jacobson radical \( J \), and let \( R^* \) be the completion of \( R \). In [2], Zariski proved that if \( R \) is a normal local integral domain, and if there is a nonzero element \( x \) in \( J \) such that \( pR^* \) is semi-prime, for every prime divisor \( p \) of \( xR \), then \( R \) is analytically unramified. In [1, p. 132] Nagata proved that if \( R \) is a semi-local integral domain, and if there is a nonzero element \( x \) in \( J \) such that, for every prime divisor \( p \) of \( xR \), \( pR^* \) is semi-prime and \( R_p \) is a valuation ring, then \( R \) is analytically unramified. (The condition \( R_p \) is a valuation ring holds if \( R \) is normal.) The main purpose of this note is to extend Nagata's result to the case where \( R \) is a semi-local ring (Theorem 1). This extension will be given after first proving a

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number of lemmas. Among these preliminary results, Lemma 3 gives a necessary and sufficient condition for \( R_p \) to be a discrete Archimedean valuation ring (where \( R \) is a Noetherian ring and \( p \) is a prime divisor of a nonzero-divisor \( b \in R \)), Corollary 2 of Lemma 5 gives a sufficient condition for a Noetherian ring to be a direct sum of normal Noetherian domains, and Lemma 6 gives a characterization of analytically unramified semi-local rings.

In Lemmas 1–4 below, \( R \) is a Noetherian ring, \( S \) is the integral closure of \( R \) in its total quotient ring, \( b \) is a nonunit in \( R \) which is not a divisor of zero, \( p \) is a prime divisor of \( bR \), and \( q \) is the isolated component of zero determined by \( p \). If \( B \) is an ideal in \( R \), then \( B'R_p \) is the ideal generated by \((B+q)/q \) in \( R_p \). Likewise, if \( c \in R \), then \( c' \) is the \( q \)-residue of \( c \).

**Lemma 1.** \((bR)_a=bS\cap R\), and an element \( c \) in \( R \) is in \((bR)_a\) if and only if \( c/b \in S \).

**Proof.** If \( c \in (bR)_a \), then \( c^n+b_1c^{n-1}+\cdots+b_n=0 \), where \( b_i \in b^iR \). Dividing this equation by \( b^n \) shows that \( c/b \in S \), so \( c/b \in bS \cap R \). If \( c/b \in bS \cap R \), then \( c/b \in S \), so \( (c/b)^n+r_1(c/b)^{n-1}+\cdots+r_n=0 \), where \( r_i \in R \). Multiplying this equation by \( b^n \) shows that \( c \in (bR)_a \), since \( c \in R \). Therefore \( bS \cap R \subseteq (bR)_a \), hence \( (bR)_a = bS \cap R \), q.e.d.

**Lemma 2.** \( R_p \) is a discrete Archimedean valuation ring if and only if \( R_p \) is normal.

**Proof.** If \( R_p \) is a valuation ring, then \( R_p \) is normal. Conversely, if \( R_p \) is normal, then \( R_p \) is a normal local integral domain (hence, the kernel of the natural homomorphism from \( R \) into \( R_p \), which is \( q \), is a prime ideal), and \( p'R_p \) is a prime divisor of \( b'R_p \). Since \( b'R_p \neq (0) \), height \( p'R_p = 1 \) hence \( R_p \) is a discrete Archimedean valuation ring \([3, \text{pp. } 276–278]\), q.e.d.

An element \( c \in R \) such that \( bR: cR = p \) is used in the next lemma. Such an element can be found as follows. Let \( p = p_1, p_2, \cdots, p_n \) be the prime divisors of \( bR \), and let \( d \) be an element in the \( p_i \)-primary component of \( bR \) (\( i = 2, \cdots, n \)) which is not in \( bR \). If \( bR: dR \neq p \), let \( e \) be an element in \( (bR: dR): pR \) which is not in \( bR: dR \), and let \( c = de \).

**Lemma 3.** Let \( c \) be an element in \( R \) such that \( bR: cR = p \). \( R_p \) is normal if and only if \( c/b \in S \).

**Proof.** Let \( R_p \) be normal. Since \( bR: cR = p \), \( b'R_p: c'R_p = p'R_p \). Therefore \( c' \in b'R_p \), so \( c'/b' \in R_p \). Hence, since \( S_{R_p} \) is contained in the integral closure of \( R_p \) in its quotient field, \( c/b \in S \). Conversely,
assume $c/b \in S$. Since $cP \subseteq bR$, $(c/b)p \subseteq R$. If $(c/b)p \subseteq p$, then $bR[(c/b) \subseteq bR[c/b] \subseteq R$, so $R[c/b]$ is contained in the finite $R$-module $(1/b)R$, hence $c/b \in S$. This is a contradiction, so $cP \nsubseteq bP$. Therefore, there are elements $d \in P$, and $x \in R$, $P \nsubseteq P$ such that $cx = bx$. Then $b'R_a = b'x'P_a = c'd'R_a \subseteq c'P_a \subseteq b'R_a$, so $c'P_a = b'R_a = c'd'R_a$. Now $c'P_a$ is not a divisor of zero in $R_a$ (since $b'x'$ is not), so $P_a = b'(c'/c')R_a$, hence $R_a$ is normal (Lemma 2, and [3, p. 277]), q.e.d.

**Lemma 4.** $(bR)_a = bR$ if and only if $R_a$ is normal, for every prime divisor $P$ of $bR$.

**Proof.** If $R_a$ is normal, for every prime divisor $P$ of $bR$, then $R_a = S_{R_a, b}$, so $P_a \cap S$ is a prime divisor of $bS$. Let $P_1, \ldots, P_a$ be the prime divisors of $bR$, and let $b_i$ be the image of $b$ in $R_a$. Then $(bR)_a = bS \cap R$ (Lemma 1) $\subseteq \left( \bigcap_{i=1}^n (b_iP_i \cap S) \right) \cap R = \left( \bigcap_{i=1}^n (b_iP_i \cap R) \right) \subseteq (bR)_a$, hence $(bR)_a = bR$. Conversely, let $(bR)_a = bR$, let $P$ be a prime divisor of $bR$, and let $c$ be an element in $R$ such that $bR = cR$. Then $c/b \in R$. If $R_a$ is not normal, then $c/b \in S$ (Lemma 3), hence $c \in (bS \cap R) = (bR)_a$ (Lemma 1). Since $(bR)_a = bR$, this is a contradiction to $c/b \in S$. Therefore $R_a$ is normal, q.e.d.

**Lemma 5.** Let $R$ be a Noetherian ring with Jacobson radical $J$, let $b$ be a nonzero element in $J$, and let $P_1, \ldots, P_n$ be the prime divisors of $bR$. If $P_{P_i}$ is a discrete Archimedian valuation ring $(i = 1, \ldots, n)$, then the isolated component of zero contained in $P_i$ is a prime ideal $q_i$, and $\cap_{i=1}^n q_i = (0)$. Moreover, $b$ is not a zero-divisor in $R$, and $(bR)_a = bR$.

**Proof.** If $P_{P_i}$ is a discrete Archimedian valuation ring, then $P_{P_i}$ is an integral domain which is not a field, so the isolated component of zero contained in $P_i$ is a prime ideal $q_i$. Since $q_i$ is the kernel of the natural homomorphism from $R$ into $P_{P_i}$, $q_i$ is contained in every $P_i$-primary ideal. Hence, since $bR = \cap_{i=1}^n (b_iP_{P_i} \cap R)$, where $b_i$ is the $q_i$-residue of $b$, and since each $P_i$ is a minimal prime divisor of $bR$, $Z = \cap_{i=1}^n q_i \subseteq bR$. Since $bP \in q_i$ $(i = 1, \ldots, n)$, $Z : bR = Z$. This implies $Z = bR \cap (Z : bR) = b(Z : bR)$. Therefore, since $b \in J$, $Z = b(Z : bR) = bZ \subseteq JZ \subseteq Z$. Hence, $Z = \cap_{i=1}^n J_iZ \subseteq \cap_{i=1}^n J_i = (0)$. Thus $b$ is not a zero-divisor, so $(bR)_a = bR$ (Lemma 4), q.e.d.

**Corollary 1.** With the same $R$ and $J$ of Lemma 5, suppose there is a nonzero nilpotent element in $R$. If $b$ is a nonzero divisor in $J$, then $(bR)_a \neq bR$.

**Proof.** If $b$ is a nonzero-divisor in $J$ such that $(bR)_a = bR$, then $R_a$ is a discrete Archimedian valuation ring, for every prime divisor $P$ of $bR$ (Lemma 4). Hence by Lemma 5, the zero ideal in $R$ is semiprime, q.e.d.
Corollary 4 below is the next result which is needed to prove Theorem 1, and it can be proved as a corollary to Lemma 5. Corollaries 1, 2, and 3, and Lemma 6 are not used in the proof of Theorem 1. They are included at this point because they are of some interest in themselves.

**Corollary 2.** Let $R$ be an integrally closed Noetherian ring, let $J$ be the Jacobson radical of $R$, and let $q_1, \ldots, q_n$ be the minimal prime divisors of zero. If there is a nonzero-divisor $b$ in $J$, then $R = \bigoplus_i R/q_i$, and $R/q_i$ is a normal Noetherian domain.

**Proof.** If $b$ is a nonzero-divisor in $J$, then $(bR)_a = bR$, since $R$ is semi-prime, and consequently the total quotient ring $Q$ of $R$ is the direct sum of $n$ fields. Since the idempotents in $Q$ are integrally dependent on $R$, they are in $R$. This, and the fact that $R$ is integrally closed, immediately imply the conclusions, q.e.d.

In Corollaries 3 and 4 and Lemmas 6 and 7, $R$ is a semi-local ring with maximal ideals $M_1, \ldots, M_d$, $T = \bigcap_i M_i$, and $R^*$ is the completion of $R$.

**Corollary 3.** Assume that no $M_i$ is a prime divisor of zero, and that $R^*$ is integrally closed. Then the completion of each $R_{M_i}$ is normal (hence $R_{M_i}$ is a normal local domain).

**Proof.** Since no $M_i$ is a prime divisor of zero, there is a nonzero-divisor $b$ in the Jacobson radical of $R^*$ [4, p. 267]. Hence by Corollary 2, $R^* = \bigoplus R^*/q_i$, where $q_i$ runs through the prime divisors of zero in $R^*$. Since the idempotents of the total quotient ring of $R^*$ are in $R^*$, no maximal ideal in $R^*$ contains more than one primed divisor of zero. Therefore, there are $d$ prime divisors of zero in $R^*$, since $R^*/q_i$ is a complete normal local domain. Let $M_i R^*$ be the maximal ideal in $R^*$ which contains $q_i$. Then it is immediately seen that $R_{M_i R^*} = R^*/q_i \bigcap R/(q_i \cap R) = R_{M_i}$. Since $R_{M_i}$ is a dense subspace of $R^*/q_i$ [4, p. 283], the completion of $R_{M_i}$ is normal. It is well known [1, p. 59] that this implies that $R_{M_i}$ is a normal local domain, q.e.d.

**Lemma 6.** Let $b$ be a nonzero-divisor in $J$, let $R^*$ be the integral closure of $R^*$ in its total quotient ring, and let $T = R^*/c \cap R^*[1/b]$. If there is an integer $n$ such that $b^n T \subseteq b R^*$, then $R$ is analytically unramified. Conversely, if $R$ is analytically unramified, then for every nonzero-divisor $c$ in $R$ there is an integer $k$ (depending on $c$) such that $c^k (R^*/c \cap R^*[1/c]) \subseteq c R^*$.

**Proof.** Since $b$ is not a divisor of zero in $R$, $b$ is not a divisor of zero in $R^*$ [4, p. 267], so $R^*[1/b]$ is contained in the total quotient
ring $Q$ of $R^*$. Let $x$ be a nilpotent element in $R^*$. Then $x/b^i \in T$, for all $i \geq 1$. Therefore, if $b^iT \subseteq bR^*$, then $x \in b^iT \subseteq b^{i+n}R^* \subseteq J^{i+n+1}R^*$, for all $i \geq n$. Since $\cap J^* = 0$, $x = 0$. Hence $R$ is analytically unramified. Conversely, let $R$ be analytically unramified and let $q_1, \ldots, q_n$ be the prime divisors of zero in $R^*$. Then $R^* = \bigoplus_i (R^*/q_i)'$, where $(R^*/q_i)'$ is the integral closure of $R^*/q_i$. Since $(R^*/q_i)'$ is a finite $R^*/q_i$-module [1, p. 112], $R^*$ is a finite $R^*$-module. Thus $R^* \cap R^*[1/c]$ is a finite $R^*$-module, for every non-zero-divisor $c$ in $R$. Hence, since every element in $R^* \cap R^*[1/c]$ can be written in the form $r/c^i$, where $r \in (cR^*)_a$, the last statement is clear, q.e.d.

**Corollary 4.** With the same notation as Lemma 6, assume $(bR^*)_a = bR^*$. Then $R$ is analytically unramified.

**Proof.** If $t \in T$, then $t = r/b^i$, where $r \in (bR^*)_a$. Since $bR^*$ and $bR^*$ have the same prime divisors, $(bR^*)_a = bR^*$ (Lemma 4). Therefore $T = R^*$, hence $bT = bR^*$, and so $R$ is analytically unramified by Lemma 6, q.e.d.

**Lemma 7.** Let $p$ be a height one prime ideal in $R$. If $R_p$ is normal, and if $pR^* = \bigcap_i p_i^*$, where each $p_i^*$ is a prime ideal in $R^*$, then each $R_p^*$ is normal, and $p^{(n)}R^* = \bigcap_i p_i^{(n)}$ (where $q^{(n)}$ is the $n$th symbolic power of a prime ideal $q$).

**Proof.** Since $R_p$ is a normal local domain which is not a field, $p$ is not a prime divisor of zero. Let $b$ be an element in $p$ such that $b'R_p = p'R_p$ ($B'R_p$ denotes the ideal in $R_p$ generated by an ideal $B$ in $R$). Then $0:bR \subseteq q$, where $q$ is the prime divisor of zero contained in $p$. Therefore, $(0:bR)R^* = 0R^* + bR^*$ [4, p. 267] $\subseteq qR^* \subseteq pR^* \subseteq p_i^*$ $(i = 1, \ldots, h)$. Fix $i$, set $p_i^* = p_i^*$, and let $q_i^*$ be a prime divisor of $0R^*$ which is contained in $p_i^*$. Then $q_i^* \cap R$ is a prime divisor of zero [4, p. 267] and is contained in $p = p^* \cap R$. Hence $q_i^* \cap R = q$. Further, since $q$ is the only $q$-primary ideal, every $q^*$-primary ideal contracts in $R$ to $q$. Hence $R_p$ is a subring of $R_p^*$, and, since $0R^* : bR^* \subseteq qR^*$, $b$ is not a zero-divisor in $R_p^*$. Since $pR^*$ is semi-prime, $b'R_p^* = p'R_p^* = p^*R_p^*$. Therefore $R_p^*$ is normal (Lemma 2 and [3, pp. 276–278]). The proof that $p^{(n)}R^* = \bigcap_i p_i^{(n)}$ is the same as that in [2]. Namely, since the result is true for $n = 1$, let $n > 1$ and assume $p^{(n-1)}R^* = \bigcap_i p_i^{*(n-1)}$. Let $c$ be an element in $bR : p$ which is not in $p$ (since $b'R_p = p'R_p$, $bR : p \subseteq p$), and let $d^* \in \bigcap_i p_i^{*(n)} \subseteq pR^*$. Since $c \in bR : pR^*$, $cd^* = br^*$, for some $r^* \in R^*$, hence by the choice of $c$ and $b$, $b'r^{*'}R_{p_i^*} = c'd^{*'}R_{p_i^*} = d^{*'}R_{p_i^*} \subseteq p_i^{*(n)}R_{p_i^*} = b'nR_{p_i^*} (i = 1, \ldots, h)$. Therefore, $r^* \in \bigcap_i p_i^{*(n-1)}$, so by induction $r^* \in p^{(n-1)}R^*$. Thus $cd^* = br^* \in p^{(n)}R^*$, hence $d^* \in p^{(n)}R^*: cR^* = (p^{(n)}: cR^*)R^*$ [4, p. 267] $= p^{(n)}R^*$, since $c \notin p$. 

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Thus $\bigcap_{i=1}^{n} p_i^{*}(n) \subseteq p^{(n)}R^*$, and since the opposite inclusion is clear, $p^{(n)}R^* = \bigcap_{i=1}^{n} p_i^{*}(n)$, q.e.d.

**Theorem 1.** Let $R$ be a semi-local ring with Jacobson radical $J$, and let $R^*$ be the completion of $R$. Assume there is a nonzero-divisor $b$ in $R$ such that $(bR)_a = bR$ and $pR^*$ is semi-prime, for every prime divisor $p$ of $bR$. Then $(bR^*)_a = bR^*$. If $b \in J$, then $R$ is analytically unramified.

**Proof.** If $b$ is a unit in $R$, then $(bR^*)_a = bR^* = R^*$. Hence assume $b$ is a nonunit in $R$, and let $p_1, \ldots, p_n$ be the prime divisors of $bR$. Since each $R_{p_i}$ is a discrete Archimedean valuation ring (Lemmas 2 and 4), every $p_i$-primary ideal is a symbolic power of $p_i$. Therefore $bR = \bigcap_{i=1}^{n} p_i^{(e_i)}$, so $bR^* = \bigcap_{i=1}^{n} (p_i^{(e_i)}R^*)$ [4, p. 269]. Fix $i$, set $p_i^{(e_i)} = p_i^{(e_i)}$, and let $p_i^*, \ldots, p_n^*$ be the prime divisors of $pR^*$. Then $p_i^{(e_i)}R^* = \bigcap_{i=1}^{n} p_i^{(e_i)}$ and each $R_{p_i^*}$ is normal (Lemma 7). Thus the prime divisors of $bR^*$ are the prime divisors of the $p_iR^* (i = 1, \ldots, h)$, hence $(bR^*)_a = bR^*$ (Lemma 4). Therefore, if $b \in J$, then by Corollary 4, $R$ is analytically unramified, q.e.d.

**Corollary 5.** Let $R, R^*$ and $b$ be as in Theorem 1, and let $S^*$ be the integral closure of $R^*$ in its total quotient ring. If there is an element $v$ in $S^*$ such that $bv \in R^*$, then $v \in R^*$.

**Proof.** $bv \in bS^* \cap R^* = (bR^*)_a = bR^*$, q.e.d.

**References**


University of California, Riverside