FINITE SUPERSOLVABLE WREATH PRODUCTS

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1. Introduction. In [1], Baumslag found all nilpotent standard restricted wreath products $A \wr B$. It is immediate that $A \wr B$ is solvable if and only if $A$ and $B$ are. In this paper we prove the following results.

**Theorem.** A finite wreath product $A \wr B$ is super solvable if and only if:

(i) $A$ is nilpotent,

(ii) either $B$ is Abelian or $B'$ is a (nontrivial) $p$-group with $A$ a $p$-group for the same prime $p$, and

(iii) for each prime $q$ dividing $o(A)$, $q \equiv 1 \pmod{m}$, where $m$ is the exponent of $B/Q$, $Q$ being the Sylow $q$-subgroup of $B$ ($Q$ is unique by (ii), and may be trivial).

**Corollary.** If $A$ and $B$ are finite and satisfy conditions (i), (ii), and (iii) above, then any extension of $A$ by $B$ is super solvable.

2. Notation and definitions. All groups considered are finite and are written multiplicatively. $E$ denotes the trivial group and 1 is used for the identity of all groups. Other notation is standard (as found, for example, in [8]).

Let $A$ and $B$ be nontrivial abstract groups, and let $F = A^B$ be the direct sum of copies of $A$ indexed by the set $B$. Explicitly, $F$ is the set of all functions from $B$ into $A$, made into a group by component-wise multiplication. For $f \in F$ and $b \in B$, define $f^b \in F$ by $f^b(y) = f(yb^{-1})$ for all $y \in B$. Then for each $b \in B$, the mapping $f \mapsto f^b$ is an automorphism of $F$, and the group of all such automorphisms is isomorphic to $B$. The (standard restricted) wreath product of $A$ and $B$, $A \wr B$, is defined to be the group generated by $F$ and $B$ with relations $b^{-1}f^b = f^b$ for all $f \in F$, $b \in B$. Here $F \triangleleft A \wr B$, and $A \wr B$ is a splitting extension of $F$ by the group of automorphisms $B$. $W$ will be used throughout to designate $A \wr B$, $F$ (called the base group) will always be as defined above, and $K$ will denote the subgroup of $W$ defined by $K = \{f \in F : \Pi_{x \in H} f(x) \in A'\}$.

We say that a group $G$ is supersolvable if and only if it has an invariant series whose factors are of prime order, that is, a series $G = G_n, G_{n-1}, \ldots, G_0 = E$ with each $G_i \triangleleft G$ and each $G_{i+1}/G_i$ of prime order.
order (also called a supersolvable series). \( G \) is nilpotent if and only if it has a series \( G = G_m, G_{m-1}, \ldots, G_0 = E \) in which each \( G_i \triangleleft G \) and each \( G_{i+1}/G_i \subseteq Z(G/G_i) \). The properties of supersolvable and nilpotent groups needed can be found in [3] or [8]. Certain of these, as well as some results on wreath products, are listed for convenience in the following lemma.

**Lemma 0.**

(0.1) \( W' = KB' \) [6, Corollary 4.5].

(0.2) If \( C \) is a subgroup of \( A \) and \( D \) a subgroup of \( B \), then \( C \wr D \) can be embedded in \( A \wr B \) [7].

(0.3) Every extension of \( A \) by \( B \) can be embedded in \( A \wr B \) [5], [7].

(0.4) \( A \wr B \) is nilpotent if and only if \( A \) and \( B \) are \( p \)-groups for the same prime \( p \) [1].

(0.5) If \( G \) is nilpotent then \( G \) is supersolvable.

(0.6) If \( G \) is supersolvable then \( G' \) is nilpotent.

(0.7) Subgroups of nilpotent (supersolvable) groups are nilpotent (supersolvable).

(0.8) \( G \) is nilpotent if and only if \( H \) a proper subgroup of \( G \) implies \( H \) a proper subgroup of \( N_0(H) \).

(0.9) \( G \) is nilpotent if and only if it is the direct product of its Sylow subgroups.

3. The proofs.

**Lemma 1.** If \( A \wr B \) is supersolvable, then \( A \) is nilpotent.

**Proof.** If \( A \) is not nilpotent, it contains a proper subgroup \( H \) with \( N_A(H) = H \) (0.8). Then \( H_1 = \{ f \in K : f(x) \in H \text{ for all } x \in B \} \) is a proper subgroup of \( K \subseteq W' \) (0.1). Suppose \( f \in N_K(H_1) \setminus H_1 \). Then there exists \( x \in B \) such that \( f(x) = a \in A \setminus H \). If we choose \( y \in B, y \neq x \), and define \( g \in H_1 \) by \( g(x) = h, g(y) = h^{-1}, \) and \( g(z) = 1 \) for \( z \neq x, y \), where \( h \) is any element of \( H \), we see that \( f^{-1}gf \in H_1 \), so that \( (f^{-1}gf)(x) = a^{-1}ha \in H \). But since \( h \) is arbitrary in \( H \) this implies \( a \in N_A(H) \setminus H \), which is impossible. Therefore \( N_K(H_1) = H_1 \), contradicting the nilpotency of \( K \) (0.8) and \( W' \) (0.7) and therefore the supersolvability of \( W \) (0.6).

**Lemma 2.** If \( A \wr B \) is supersolvable, then \( B \) is Abelian or \( B' \) is a (nontrivial) \( p \)-group and \( A \) is a \( p \)-group for the same prime \( p \).

**Proof.** If the lemma is false, then there are primes \( p \mid o(B'), q \mid o(A), p \neq q \). Take \( a \in A \) with \( o(a) = q \) and \( b \in B' \) with \( o(b) = p \). Choose arbitrary distinct elements \( x, y \in B \), and define \( f \in F \) by \( f(x) = a, f(y) = a^{-1}, f(z) = 1 \) for \( z \neq x, y \). Then \( f \in K \subseteq W' \), \( b \in W' \), \( o(f) = q, o(b) = p \), but \( fb \neq bf \). (If \( a = a^{-1} \) for all \( a \in A \), then \( o(B) \geq 2 \) and
x and y can be chosen so that we have \( f_b \neq bf \). Therefore \( W' \) is not nilpotent (0.9) and so \( W \) is not supersolvable (0.6).

Lemma 3. \( A \wr B \) is supersolvable if and only if \( A \) is nilpotent and \( J_p \wr B \) is supersolvable for each \( p \) dividing \( o(A) \), where \( J_p \) denotes the cyclic group of order \( p \).

Proof. If \( A \) is not nilpotent, \( W \) is not supersolvable by Lemma 1, while if \( p \mid o(A) \) and \( J_p \wr B \) is not supersolvable, then \( A \wr B \) is not supersolvable by (0.2) and (0.7).

For sufficiency, induct on \( o(A) \). There is a subgroup \( H \) of \( Z(A) \) of prime order. Since \( H \wr B \) is supersolvable, there is an invariant series of \( W \) up through \( H^B \) with factors of prime order. Then \( W/H \cong (A/H) \wr B \) is supersolvable by induction, and therefore \( W \) is also supersolvable.

Lemma 4. If \( p \nmid n \), \( p \) prime, then \( J_p \wr J_n \) is supersolvable if and only if \( p \equiv 1 \pmod{n} \).

Proof. Since \( J_n \) is supersolvable and \( W/F \cong J_n \), and \( F \) is Abelian, it is necessary and sufficient to show that \( F \) has a series \( F = F_n, F_{n-1}, \ldots, F_0 = E \) with each \( o(F_i/F_{i+1}) = p \) and each \( F_i \) invariant under the action of \( B = J_n \). Here we may view \( F \) as a vector space of dimension \( n \) over \( GF(p) \), a field with \( p \) elements, acted on by the cyclic linear transformation \( b \) induced by \( b \), where \( \langle b \rangle = B = J_n \). The minimum polynomial of \( b \) is \( \lambda^n - 1 \), and the subspaces invariant under \( b \) may be put in one-one correspondence with the factors of \( \lambda^n - 1 \) which have leading coefficient 1, with one such subspace containing another if and only if the factor corresponding to the first divides the factor corresponding to the second [4, p. 129]. Therefore \( J_p \wr J_n \) is supersolvable if and only if \( \lambda^n - 1 \) splits into linear factors over \( GF(p) \). Since \( p \nmid n \), \( \lambda^n - 1 \) has no repeated roots in \( GF(p) \), and so \( \lambda^n - 1 \) splits into linear factors if and only if \( n \mid (p-1) \), for the roots of \( \lambda^n - 1 \) form a subgroup of the multiplicative group of \( GF(p) \).

Lemma 5. If \( B \) is Abelian and \( p \nmid o(B) \), then \( J_p \wr B \) is supersolvable if and only if \( p \equiv 1 \pmod{m} \), where \( m = \text{Exp } B \).

Proof. If \( p \equiv 1 \pmod{m} \), we may apply (0.2), (0.7), and Lemma 4 to conclude that \( J_p \wr B \) is not supersolvable since \( J_p \wr J_m \) is not. As in the proof of Lemma 4, it is now sufficient to show that if \( p \equiv 1 \pmod{m} \) then \( F \) has a series \( F = F_n, F_{n-1}, \ldots, F_0 = E \) with each \( o(F_i/F_{i+1}) = p \) and each \( F_i \) invariant under the action of \( B \), where in the present case \( n = o(B) \). We interpret \( B \) as a group acting on \( F \), a vector space of dimension \( n \) over \( GF(p) \), and apply Maschke's
Theorem [2, pp. 40–41] to conclude that $F$ is a direct sum of irreducible $B$-subspaces. If all of these irreducible $B$-subspaces are of dimension one, it is clear how to construct the sequence $F = F_n, F_{n-1}, \ldots, F_0 = E$. However, since $p \mid \text{Exp } B$, it follows [2, Theorem 9.10, p. 37] that $B$ has $o(B)$ distinct one-dimensional $GF(p)$-representations, and therefore all irreducible $B$-subspaces are one-dimensional [2, p. 213]. (Since $p \equiv 1 \pmod{m}$, $\lambda^{m} - 1$ splits into linear factors over $GF(p)$, and so in the notation of [2], $\sqrt{1} \subseteq GF(p)$.)

**Lemma 6.** If $P$ is a Sylow $p$-subgroup of $B$ containing $B'$, then $J_p \ltimes B$ is supersolvable if and only if $p \equiv 1 \pmod{m}$, where $m = \text{Exp } (B/P)$.

**Proof.** Suppose $J_p \ltimes B$ is supersolvable. Since $P \triangleleft B$, $P$ has a complement $Q$ in $B$ by Schur's splitting theorem [8, 9.3.6]. Then $J_p \ltimes Q$ is supersolvable by (0.2) and (0.7), and hence $p \equiv 1 \pmod{m}$ by Lemma 5. Now assume $p \equiv 1 \pmod{m}$. We can use the fact that $P \triangleleft B$ to conclude that $FP \triangleleft W$, and thereby think of $W$ as an extension of $FP$ by $B/P$. Then Lemmas 3 and 5 and (0.3) and (0.7) apply to yield that $J_p \ltimes B$ is supersolvable.

**Proof of Theorem.** If $A \ltimes B$ is supersolvable, the necessity of (i) follows from Lemma 1, that of (ii) from Lemma 2, and that of (iii) from Lemma 3 and Lemma 6. Sufficiency is shown by Lemmas 3, 5, 6, and (0.4).

**Proof of Corollary.** Apply the Theorem, (0.3), and (0.7).

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**References**


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