BIBLIOGRAPHY


UNIVERSITY OF MICHIGAN

---

KIRZBRAUN'S THEOREM AND KOLMOGOROV'S PRINCIPLE

EDWARD SILVERMAN

Let $B$ be a Banach space. A distance function $p$ on $B$ is a non-negative valued function which is continuous, positively homogeneous of degree one and subadditive. If $A$ is a set and if $x$ and $y$ map $A$ into $B$ then we write $xpy$ if $p(x(a) - x(b)) \leq p(y(a) - y(b))$ for all $a, b \in A$. If $A$ is a $k$-cell, if $B$ is Euclidean space, if $p$ is the norm and if $L$ is Lebesgue area, then Kolmogorov’s Principle, K.P., asserts that $Px^y$ if $xpy$ [H.M.]. Lebesgue area is a parametric integral of the type considered by McShane [M], for smooth enough maps. In this paper we consider other such integrals, not necessarily symmetric, for which a type of K.P. holds. We conclude with a minor application to a Plateau problem.

The proof of K.P. follows from

**Kirzbraun’s Theorem.** If $A \subseteq \mathbb{E}^n$ and $t: A \rightarrow \mathbb{E}^n$ is Lipschitzian, then there exists an extension $T$ of $t$, $T: \mathbb{E}^n \rightarrow \mathbb{E}^n$, and $T$ is Lipschitzian with the same constant as $t$ [S].

The proof of the version of K.P. in which we are interested depends upon an embedding of $\mathbb{E}^n$ in $m$, the space of bounded sequences [B],

---

Received by the editors January 11, 1965.

1 This research was supported in part by National Science Foundation Grant No. GP634.
and an extension theorem, resembling that of Kirzbraun, for \( m \).

Let \( \alpha \) be the distance function on \( m \) defined by \( \alpha(a) = \max \{ \sup a^i, 0 \} \). In a manner to be made precise in the Embedding Theorem, \( \alpha \) is universal as a distance function.

Let \( B \) be a separable Banach space and \( p \) be a distance function on \( B \). Let \( \{ b_i \} \) be dense on \( \partial K \) where \( K = \{ b \in B \mid p(b) \leq 1 \} \). There exist, by the Hahn-Banach Theorem \([B]\), \( f_i \in B^* \) such that \( f_i(b) = 1 \) and \( f_i(b) \leq p(b) \) for all \( b \in B \). Since \( p \) is continuous there exists \( N' > 0 \) such that \( \|f_i\| \leq N' \) for all \( i \).

**Embedding Theorem** \([S3]\). Let \( V_b = \{ f_i(b) \} \). Then \( V \in L(B, m) \) and \( p = \alpha V \).

The proof is almost immediate.

Let \( N \) be the set of natural numbers. If \( k \in N \), then \( \Lambda^k m \) is the space of all bounded real-valued anti-symmetric functions on \( N^k \) with the sup norm.

If \( a = (a_1, \cdots, a_k) \in m^k \) let \( U_a \in L(E^k, m) \) and \( \Lambda a = a_1 \Lambda \cdots \Lambda a_k \in \Lambda^k m \) be defined by \( U_a h = \sum_{i=1}^k h_i a_i \) for all \( h = (h_1, \cdots, h_k) \in E^k \), and

\[
(\Lambda a)(n_1, \cdots, n_k) = \det \begin{bmatrix} a_1 & \cdots & a_1 \\ \vdots & \cdots & \vdots \\ a_k & \cdots & a_k \\ \end{bmatrix} 
\]

where, of course, \( a_i = (a_1^i, a_2^i, \cdots) \). Furthermore, if \( \xi = (\xi_1, \cdots, \xi_k) \in m^{*k} \) then

\[
[\Lambda a, \xi_1 \Lambda \cdots \Lambda \xi_k] = \det \begin{bmatrix} \xi_1(a_1) & \cdots & \xi_k(a_1) \\ \vdots & \cdots & \vdots \\ \xi_1(a_k) & \cdots & \xi_k(a_k) \\ \end{bmatrix} .
\]

If \( E^k \subset m \) and \( a_1, \cdots, a_k \in E^k \), then \( \|\Lambda a\| \) is the volume of the parallelepiped spanned by \( a_1, \cdots, a_k \). If similarly, \( b \in (E^k)^k \subset m^k \) then \( \|\Lambda a\| \leq \|\Lambda b\| \) if \( U_a \preceq U_b \), and this fact is vital for the validity of K.P. In general, we write \( a \prec b \) if \( U_a \preceq U_b \).

Let \( M \) be the set of all distance functions on \( \Lambda^k m \) and let \( M' = \{ f \in M \mid f(\Lambda a) \leq f(\Lambda b) \text{ whenever } a \prec b \} \).

Let \( Q \) be a \( k \)-cell. If \( x \in C(Q, m) \) is Lipschitzian, then \( dx = \{ \partial x^i/\partial u^i \} \) exists almost everywhere. We write \( dx \) for \( (d_1 x, \cdots, d_k x) \in m^k \) and, as above, \( \Delta dx \) for \( d_1 x \Lambda \cdots \Lambda d_k x \).

Suppose that \( B \) is a Banach space contained in \( m \). Then we can identify \( \Lambda^k B \) with the appropriate subspace of \( \Lambda^k m \). Let \( Q \) be a \( k \)-cell and \( P(Q, B) \) be the subset of \( C(Q, B) \) consisting of quasilinear func-
tions. Then \( P(Q, B) \) is dense in \( C(Q, B) \). If \( z \in P(Q, B) \) and \( f \in M \) then we define \( \varepsilon_f z = \sum f(\Delta z) \cdot \operatorname{vol} \Delta \) where the summation is taken over the oriented simplexes, \( \Delta \), of linearity of \( z \). Let \( M'' = \{ f \in M' \mid \varepsilon_f \) is lower semi-continuous on \( P(Q, B) \} \). If \( f \in M'' \) then the Fréchet extension, \( L_{f,B} \), of \( \varepsilon_f \) is the Lebesgue area: \( L_{f,B} = \lim \inf_{n \to \infty} \varepsilon_f z \) for each \( x \in C(Q, B) \) where \( z \in P(Q, B) \). Let us write \( L_f \) for \( L_{f,m} \). We will make use of the fact, proved like the two-dimensional case in [S3], that \( L_{f,B} = \int f(\Delta z) = L_f x \) for \( x \) in a subset of \( C(Q, B) \) which contains, in particular, all Lipschitzian maps.

It is obvious that \( L_f \leq L_{f,B} \), but not at all evident that the equality holds, though this known for \( k = 2 \) and for a wide class of functions for \( k > 2 \).

Let \( e \) be the identity mapping on \( m \).

**Extension Theorem.** Let \( \sigma > 0, A \subset m \) and \( t : A \to m \) with \( t \alpha(\sigma \epsilon | A) \), i.e., \( \alpha(ta - tb) \leq \sigma \alpha(a - b) \) for all \( a, b \in A \). Then there exists an extension \( T \) of \( t \) such that \( t \alpha(\sigma \epsilon) \). Let \( \psi_i(a, c) = (ta)_i + \sigma \alpha(c - a) \) for all \( a, c \in A \) and \( c \in m \), and let \( T_i(c) = \inf \{ \psi_i(a, c) \mid a \in A \} \). Then we can let \( T \epsilon = \{ T_i(c) \} \).

The proof is like that of the corresponding theorem in [S3]. First suppose that \( c \in A \). Then \( \psi_i(a, c) - \psi_i(c, c) = (ta)_i + \sigma \alpha(c - a) - (tc)_i \) \( \geq \sigma \alpha(c - a) - \alpha(tc - ta) \geq 0 \) and so \( T | A = t \). If \( b \in m \) then \( \psi_i(a, b) - \psi_i(a, c) = \sigma[\alpha(b - a) - \alpha(c - a)] \leq \sigma \alpha(b - c) \) and so \( \psi_i(a, b) - \psi_i(a, c) \leq \sigma \| b - c \| \). Thus \( \| T_i(b) \| \leq \| tc \| + \sigma \| b - c \| \) and \( T b \in m \) for all \( b \). If \( d \in m \) then \( [T_i(b) - T_i(d)] \leq \sup \{ \sigma[\alpha(b - a) - \alpha(d - a)] \leq \sigma \alpha(b - d) \), and the proof is complete.

Let \( \pi_n a = b \) where \( b^i = a^i \) if \( i \leq n \) and \( b^i = 0 \) for \( i > n \).

It is clear that \( x_n \to y \), in \( C(Q, m) \), if and only if

\[
\max \{ \alpha(x_n(p) - y(p)) \mid p \in Q \} \to 0 \quad \text{and} \quad \max \{ \alpha(y(p) - x_n(p)) \mid p \in Q \} \to 0.
\]

Hence \( T x_n \to T y \) whenever \( x_n \to y \) and \( T \alpha \).

**Kolmogorov’s Principle.** If \( x, y \in C(Q, m) \), if \( x \alpha y \), and if \( f \in M'' \), then \( L_f x \leq L_f y \).

Let \( z \in P(Q, m) \) and \( T \alpha \). Let \( w = T z \) and \( w_n = \pi_n w \). Then \( w \) is Lipschitzian. Hence for almost all \( p \in Q \), all \( h \in R^k \) and \( s > 0 \), \( \alpha(s \Delta w_n(p) \cdot h + o(s)) = \alpha((w_n(p + sh) - w_n(p)) \leq \sigma \alpha(s \Delta z(p) \cdot h) \). It follows that \( d w(p) \prec d \alpha(p \cdot h) \) and, since \( f \in M'' \), \( L_f(T z) = \int f(\Delta w) \leq \int f(\Delta z) = \varepsilon_f z \).

If we let \( T y = x \) then \( t \alpha(\epsilon \mid \text{range} y) \) and, by the Extension Theorem, there exists \( T \alpha \) with \( T y = x \). Let \( z_n \to y \) with \( z_n \in P(Q, m) \) and \( \varepsilon_f z_n \to L_f y \). Then \( T z_n \to x \) and \( L_f x \leq \lim \inf L_f(T z_n) \leq \lim \varepsilon_f z_n = L_f y \).

If seems appropriate to show that there exist \( A \in M'' \) such that \( A \) is not symmetric.
If \( a \in m^k \), then \( R_a \), the range of \( U_a \), is the vector subspace of \( m \) spanned by the components of \( a \) and \( R_a^* \) is the space of linear functionals over \( R_a \). If \( \xi \in R_a^* \), let \( N_a(\xi) = \inf \{ K | K\alpha(c) \geq \xi(c) \text{ for all } c \in R_a \} \). Let \( A(\Delta a) = \sup \{ [\Delta a, \xi_1 \Delta \xi_2 \Lambda \cdots \Lambda \xi_k^k] | N_a(\xi_i) \leq 1 \} \). Suppose that \( a < b \) and \( T \in L(R_b, R_a) \) is defined by \( U_a = T \circ U_b \). Let \( T^* \in L(R_a^*, R_b^* \) be defined by \( T^* \xi = \xi \circ T \). If \( K\alpha \circ U_a \geq \xi \circ U_a \), then \( K\alpha \circ U_b \geq T^* \xi \circ U_b \) and \( N_b(T^* \xi \leq N_a(\xi) \). Thus

\[
A(\Delta a) = \sup \{ [\Delta b, T^* \xi_1 \Lambda \cdots \Lambda T^* \xi_k^k] | N_a(\xi_i) \leq 1 \}
\]

\[
\leq \sup \{ [\Delta b, \eta_1 \Lambda \cdots \Lambda \eta_k] | N_b(\eta_i) \leq 1 \} = A(\Delta b).
\]

For each \( \gamma \in \Delta^k m \) we set

\[
A(\gamma) = \inf \left\{ \sum_{i=1}^p A(\Delta a_i) \left| \sum_{i=1}^p (\Delta a_i)(n) \geq \gamma(n) \text{ for all } n \in N^k \right. \right\}.
\]

Then \( A \in M' \). It can be shown, by making suitable modifications in the argument of [S3], that \( A \in M'' \).

We conclude with an interesting, though trivial, application. If \( f \in M'' \) and \( \phi \in C(\partial Q, m) \) and \( B(\phi) = \inf \{ L_f x | x \in C(Q, m) \text{ and } x \big| \partial Q = \phi \} \). Then \( B(\phi) \leq B(\phi') \text{ if } \phi \leq \phi' \). The proof goes as follows: There exists \( T \alpha \) with \( \phi = T \phi' \). Let \( y \in C(Q, m) \) with \( y \big| \partial Q = \phi' \). Then \( T y \in C(Q, m) \) and \( T y \big| \partial Q = \phi \). Hence \( B(\phi) \leq L_f (T y) \leq L_f y \).

**References**


