NILPOTENT ELEMENTS IN RINGS OF
INTEGRAL REPRESENTATIONS

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1. Introduction. Let $G$ be a finite group, and let $R$ be a discrete valuation ring of characteristic zero, with maximal ideal $P = \pi R$, and whose residue class field $\overline{R} = R/P$ has characteristic $p \neq 0$. By an $RG$-module we mean always a left $RG$-module which is finitely generated over $R$, though not necessarily $R$-torsionfree. Assume that the Krull-Schmidt theorem is valid for $RG$-modules; this is certainly the case when $R$ is complete, or when $R$ is a valuation ring in an algebraic number field which is a splitting field for $G$.

In a recent paper [4] we introduced the integral representation ring, denoted by $A(RG)$, defined as the additive group generated by the symbols $\{ M \}$, one for each isomorphism class of $R$-torsionfree $RG$-modules, with relations $\{ M \otimes N \} = \{ M \} + \{ N \}$. Multiplication in $A(RG)$ is defined by taking tensor products of modules.

The question arises as to whether the commutative ring $A(RG)$ contains any nonzero nilpotent elements. This is of special interest in view of recent results of Green [2] and O'Reilly [3], who showed that if $k$ is a field of characteristic $p$, and if $G$ has a cyclic $p$-Sylow subgroup, then $A(kG)$ has no nonzero nilpotent elements.

In contradistinction to this, we proved in [4]:

**Theorem 1.** Let $G^*$ be a cyclic group of order $n$, and suppose that the Krull-Schmidt theorem holds for $RG^*$-modules. Assume that $n \in P^2$, and if $2 \in P$ assume further that $n \in 2P$. Then $A(RG^*)$ contains at least one nonzero nilpotent element.

The aim of the present note is to establish the following generalization.

**Theorem 2.** Suppose that the group $G$ contains a cyclic subgroup $G^*$ satisfying the hypotheses of Theorem 1, and assume that the Krull-Schmidt theorem holds for $RG$-modules. Then $A(RG)$ contains at least one nonzero nilpotent element.

We shall use the following notation. For $M$ an $RG$-module, set $\overline{M} = M/PM$. Denote by $M_H$ the $RH$-module obtained from $M$ by
restriction of operators, where $H$ is a subgroup of $G$. For $N$ an $RH$-module, let $N^G$ denote the induced $RG$-module defined by

$$N^G = RG \otimes_{RH} N.$$  

The trivial $RG$-module is $R$ itself, on which each $g \in G$ acts as identity operator. If $M$, $N$ are $RG$-modules, the notation $M \mid N$ means that $M$ is isomorphic to an $RG$-direct summand of $N$.

As general reference for the techniques and definitions used in this note, we refer the reader to [1].

2. Preliminaries to the proof. Suppose hereafter that the hypotheses of Theorem 2 are satisfied, so that $G$ contains a cyclic subgroup $G^*$ of order $n$. If $p$ is the unique rational prime contained in $P$, then the assumptions about $n$ readily imply that $p^e \mid n$, where

$$e = \begin{cases} 2, & p = 2, \\ 1, & p \text{ odd, } p \in P^2, \\ 2, & p \text{ odd, } p \in P^2. \end{cases}$$

Hence $G^*$ contains a cyclic subgroup $H$ of order $p^e$. Since the Krull-Schmidt theorem is assumed valid for $RG^*$-modules, it also holds for $RH$-modules. Note that $p^e \in P^*$, and if $p = 2$, then $p^e \in 2P$.

Let $I$ denote the augmentation ideal of $RH$, so that

$$I = \sum_{h \in H} R(h - 1).$$

Then

$$\bar{I} = \sum_{h \in H} \overline{R}(h - 1) \cong \overline{R}[x]/(x - 1)^{p^e-1},$$

where the generator of the cyclic group $H$ acts on the right-hand module as multiplication by $x$. This shows that $\bar{I}$ is indecomposable, whence so is $I$.

We shall show next that if $K$ is a proper subgroup of $H$, and $\overline{M}$ is any $\overline{RK}$-module, then $\bar{I} \mid \overline{M}^H$ is impossible. For if such a relation were true, we could assume without loss of generality that $\overline{M}$ is an indecomposable $\overline{RK}$-module. Since $K$ is cyclic, the indecomposable $\overline{RK}$-modules may be listed explicitly, and have $K$-dimensions $1, 2, \cdots, [K:1]$. If $\overline{M}$ has dimension $[K:1]$, then $\overline{M} = \overline{RK}$, and in that case $\bar{I} \mid \overline{RH}$. This is impossible since $\overline{RH}$ is indecomposable. On the other hand, if $\dim \overline{M} < [K:1]$, then $\dim \overline{M}^H = [H:K] \cdot \dim \overline{M} < p^e - 1 = \dim \bar{I}$, so also in this case $\bar{I}$ cannot be a direct summand of $\overline{M}^H$.

As in [4], define the $RH$-modules $X$ and $Y$ by
Then $X$ is a nonsplit extension of the factor module $R$ (with trivial action of $H$) by the submodule $I$, and hence $X$ is indecomposable. On the other hand, we showed in [4] that $Y$ is an extension of $R$ by a submodule $J$, where $\overline{R}|\overline{J}$. From these facts we were able to conclude that

$$X \cong Y \cong \overline{R} \oplus I,$$

and hence $X$ is not isomorphic to $Y$.

Furthermore, there exist exact sequences

$$0 \to X \to RH \to \overline{R} \to 0$$

(1)

$$0 \to Y \to RH \to I \to 0.$$

Let $r$ be a positive integer such that $p|r$, and let $M$ be any $RH$-module. Define a new $RH$-module $M_r$ consisting of the same elements as $M$, but with a different action of $H$, namely, an element $h \in H$ acts on $M$, in the same way that $h^r$ acts on the original module $M$. Clearly, if $M$ is an extension of $A$ by $B$, then $M_r$ is an extension of $A_r$ by $B_r$.

Further, if $R|\overline{M}$ then also $R|\overline{M_r}$.

We claim that $X \cong Y_r$ is impossible. To prove this, note first of all that $Y_r$ is an extension of $R$ by $J_r$. If $X \cong Y_r$ then $I \cong J_r$, and so $\overline{I} \cong \overline{J_r}$. This cannot hold true because $\overline{R}|\overline{J}$, so that $\overline{R}|\overline{J_r}$, while on the other hand $\overline{R}|\overline{I}$. We have thus established our claim.

3. Proof of the main result. Keeping the notation of the preceding section, we are now ready to prove Theorem 2. Let us define $U = X^g$, $V = Y^g$. Since $(RH)^g \cong RG$, from the exact sequences in (1) we obtain a new pair of exact sequences

$$0 \to U \to RG \to \overline{R}^g \to 0,$$

$$0 \to V \to RG \to I^g \to 0.$$

Since $X \cong \overline{V}$ we conclude that $\overline{U} \cong \overline{V}$.

For any $RG$-module $M$ of $R$-rank $m$, there is an exact sequence

$$0 \to U \otimes_R M \to RG \oplus \cdots \oplus RG \to \overline{R}^g \otimes_{RG} M \to 0,$$

where $m$ summands occur in the center module. This implies by Schanuel's Lemma that the module $U \otimes_R M$ depends only upon $\overline{M}$. Hence we may conclude that

$$U \otimes_R U \cong U \otimes_R V \cong V \otimes_R U \cong V \otimes_R V,$$

and therefore that $\{U\} - \{V\}$ has square zero in $A(RG)$. 

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In order to complete the proof of Theorem 2, it suffices to show that 
\( \{U\} - \{V\} \) is nonzero, that is, that \( U \) and \( V \) are not isomorphic. Suppose that \( U \cong \cong V \), so that \( X^g \cong Y^g \). Then

\[ (X^g)_H \cong (Y^g)_H \] as \( RH \)-modules.

However, \( X \mid (X^g)_H \), and so also \( X \mid (Y^g)_H \). Let us apply Mackey's Subgroup Theorem to the module \( (Y^g)_H \). This yields

\[ (Y^g)_H \cong \sum_g (g \otimes Y)_K^H. \]

In this formula, \( g \) ranges over a full set of representatives of the \( (H, H) \)-double cosets of \( G \). For each such \( g \), \( K \) is the subgroup of \( H \) given by \( K = H \cap gHg^{-1} \). The \( RK \)-module \( g \otimes Y \) is a subspace of \( RG \otimes Y \), and the action of \( K \) on \( g \otimes Y \) is given by

\[ ghg^{-1}(g \otimes y) = g \otimes hy, \quad h \in H, \quad y \in Y. \]

Since \( X \mid (Y^g)_H \) and \( X \) is indecomposable, we conclude that \( X \mid (g \otimes Y)_K^H \) for some \( g \), and hence that \( X \mid (g \otimes Y)_K^H \). By the remarks in §2, this cannot occur if \( K \) is a proper subgroup of \( H \). On the other hand, suppose that \( K = H \), so that \( gHg^{-1} = H \). If \( h \) is a generator of the cyclic group \( H \), we may write \( g^{-1}hg = hr \), where \( p \mid r \). Then the \( RH \)-modules \( g \otimes Y \) and \( Y_r \) are isomorphic, and if \( X \mid (g \otimes Y) \), then \( X \cong Y_r \). This is impossible by the results of §2. We have thus shown that \( U \) and \( V \) are nonisomorphic, which completes the proof of the theorem.

**Corollary.** Let \( R_0 \) be a valuation ring in an algebraic number field, with maximal ideal \( P_0 \). Suppose that \( G \) contains a cyclic sub-group \( G^* \) of order \( n \), where \( n \in P_0^2 \), and if \( 2 \in P_0 \), assume further that \( n \in 2P_0 \). Then \( A(R_0G) \) contains at least one nonzero nilpotent element.

**Proof.** Even though the Krull-Schmidt theorem may not hold for \( R_0G \)-modules, we may nevertheless form the representation ring \( A(R_0G) \) defined as above. If \( L \) and \( L' \) are \( R_0G \)-modules, it is easily seen that \( \{L\} = \{L'\} \) in \( A(R_0G) \) if and only if there exists an \( R_0G \)-module \( M \) such that \( L \oplus M \cong L' \oplus M \).

The construction given in Theorem 2, with \( R \) replaced by \( R_0 \), yields a pair of \( R_0G \)-modules \( U_0, V_0 \) such that \( U = R \otimes U_0, V = R \otimes V_0 \). However, the map \( A(R_0G) \rightarrow A(RG) \) defined by \( L \mapsto R \otimes R_0 \otimes L \) is a monomorphism. Therefore \( \{U_0\} - \{V_0\} \) is a nonzero nilpotent element of \( A(R_0G) \), and the Corollary is proved.
NOTE ON ANALYTICALLY UNRAMIFIED
SEMI-LOCAL RINGS

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All rings in this paper are assumed to be commutative rings with a unit element. If $B$ is an ideal in a ring $R$, the integral closure $B_a$ of $B$ is the set of elements $x$ in $R$ such that $x$ satisfies an equation of the form $x^n + b_1x^{n-1} + \cdots + b_n = 0$, where $b_i \in B_i$ $(i = 1, \cdots, n)$. An ideal $B$ in $R$ is semi-prime in case $B$ is an intersection of prime ideals. If $R$ is an integral domain, then $R$ is normal in case $R$ is integrally closed in its quotient field. If $P$ is a semi-local (Noetherian) ring, then $P$ is analytically unramified in case the completion of $P$ (with respect to the powers of the Jacobson radical of $R$) contains no nonzero nilpotent elements.

Let $R$ be a semi-local ring with Jacobson radical $J$, and let $R^*$ be the completion of $R$. In [2], Zariski proved that if $R$ is a normal local integral domain, and if there is a nonzero element $x$ in $J$ such that $pR^*$ is semi-prime, for every prime divisor $p$ of $xR$, then $R$ is analytically unramified. In [1, p. 132] Nagata proved that if $R$ is a semi-local integral domain, and if there is a nonzero element $x$ in $J$ such that, for every prime divisor $p$ of $xR$, $pR^*$ is semi-prime and $R_p$ is a valuation ring, then $R$ is analytically unramified. (The condition $R_p$ is a valuation ring holds if $R$ is normal.) The main purpose of this note is to extend Nagata's result to the case where $R$ is a semi-local ring (Theorem 1). This extension will be given after first proving a

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