INCLUSION RELATIONS AMONG ORLICZ SPACES

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This paper contains two results; the first extends to a wide class of Orlicz spaces the statement, due to Krasnosel'skii and Rutickii [1, p. 60], that \( L_1 \) is the union of the Orlicz spaces which it contains properly; the second shows that for a wide class of spaces this is not true, i.e. there exists a set of Orlicz spaces no one of which is the union of the Orlicz spaces it contains properly. Here the Orlicz spaces are defined on \([0, 1]\) which is given Lebesgue measure \( \mu \).

1. We give in this section several definitions together with some elementary results about Orlicz spaces and convex functions.

Let \( \mathcal{C} \) be the set of convex symmetric functions \( \Phi: (-\infty, \infty) \to [0, \infty) \) such that \( \Phi(0) = 0 \), \( \lim_{s \to 0} \Phi(s)/s = 0 \) and \( \lim_{s \to \infty} \Phi(s) = \infty \). If \( \Phi \) and \( \Omega \) are two elements of \( \mathcal{C} \), we say \( \Phi \preceq \Omega \) if there exist constants \( c \) and \( s_0 \) such that \( \Phi(s) \leq \Omega(cs) \) for all \( s \geq s_0 \). We say \( \Phi \simeq \Omega \) if \( \Phi \preceq \Omega \) and \( \Omega \preceq \Phi \); we say \( \Phi \preceq \Omega \) if \( \Phi \preceq \Omega \) but \( \Omega \npreceq \Phi \). If \( \Phi_1 \simeq \Phi_2 \) and \( \Phi_1 \preceq \Phi_2 \) (\( \Phi_1 < \Omega_1 \)) then \( \Phi_2 \preceq \Omega_2 \) (\( \Phi_2 < \Omega_2 \)).

If \( \Phi \in \mathcal{C} \), then there exists a nondecreasing function \( \phi: [0, \infty) \to [0, \infty) \) such that \( \phi(0) = 0 \), \( \lim_{s \to \infty} \phi(s) = \infty \) and

\[
\Phi(s) = \int_0^{s/\phi} \phi(t) \, dt
\]

(see [1, p. 5]). This representation for \( \Phi \) yields easily the two following inequalities:

\[
(1) \quad \frac{s}{2} \phi\left(\frac{s}{2}\right) \leq \Phi(s) \leq \phi(s),
\]

\[
(2) \quad 2\Phi(s) \leq \Phi(2s).
\]

Let

\[
L_\Phi^* = \{f \in L_1: \Phi(cf) \in L_1 \text{ for some positive real number } c\}.
\]

The set \( L_\Phi^* \) is called an Orlicz space. It has a unique uniformity which is compatible with the order relation. Since this uniformity does not intervene in what follows, we do not give its definition; for this see [1, p. 69].

Received by the editors April 26, 1965.

1 This research was supported by N.S.F. grant no. GP2491.
The order relations among the elements of \( C \) give rise to order relations among the \( L_\Phi^* \) as follows:

(a) \( \Phi \leq \Omega \) implies \( L_\Phi^* \subset L_\Omega^* \).
(b) \( \Phi \sim \Omega \) implies \( L_\Phi^* = L_\Omega^* \).
(c) \( \Phi < \Omega \) implies \( L_\Phi^* \subset L_\Omega^* \) but \( L_\Phi^* \not\subset L_\Omega^* \).

Statements (a) and (b) are direct consequences of the definitions; while (c) is a special case of

(d) \( \limsup_{s \to \infty} \Omega(\alpha s)/\Phi(s) = \infty \) for all \( \alpha > 0 \) implies there exists \( f \in L_\Phi^* \) such that \( f \notin L_\Omega^* \).

**Proof.** Let \( E_{ij} \) be a pairwise disjoint double sequence of intervals in \([0, 1]\) such that \( \mu(E_{ij}) \neq 0 \), \( i, j = 1, 2, \ldots \). For each pair of natural numbers \( (n, i) \) there exists a number \( s_{ni} > 0 \) such that \( s > s_{ni} \) implies \( \Phi(s)\mu(E_{ni}) > 2^{-i}n^{-2} \). There exist numbers \( c_{ni} > s_{ni} \) such that \( \Omega(c_{ni})/n > n^2\Phi(c_{ni}) \).

Let \( E_{ni} \) be a nonempty subinterval of \( E_{ni} \) such that \( \Phi(c_{ni})\mu(E_{ni}) = 2^{-i}n^{-2} \) and define

\[
f(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij} X_{E_{ij}}(x).
\]

It is easy to show that \( f \in L_\Phi^* \) but \( f \notin L_\Omega^* \).

(e) One can use (a), (b) and (d) to show that \( \Phi < \Omega \) if \( L_\Phi^* \) is a proper subset of \( L_\Omega^* \).

2. We say that \( \Phi \in C \) satisfies * if

\[
\limsup_{s \to \infty} \frac{\Phi(2s)}{\Phi(s)} < \infty
\]

and we say it satisfies ** if

\[
\liminf_{s \to \infty} \frac{\Phi(2s)}{s\Phi(s)} > 0.
\]

These conditions are similar to the \( \Delta_2 \) and \( \Delta_3 \) conditioned in [1]. A function \( \Phi \) which satisfies * grows less rapidly than some power and in addition it grows regularly; while a function \( \Phi \) which satisfies ** grows like exp. It follows that \( C \) contains functions which satisfy neither * nor **.

**Theorem 1.** Suppose \( \Phi \in C \) and \( \Phi \) satisfies *; then, \( L_\Phi^* \) is the union of the Orlicz spaces it contains properly.

**Proof.** Let \( f \in L_\Phi^* \). We will prove there exists \( \Omega \) in \( C \) such that \( \Phi < \Omega \) and such that \( f \in L_\Omega^* \). If \( f \in L_\infty \), we are finished because \( L_\infty \subset L_\Phi^* \) properly. We will assume that \( f(x) \geq 0 \) a.e.; this is in order because
Let \( f \in L^*_\Phi \) if \( |f| \in L^*_\Phi \). Let \( c \) be a positive real number such that \( \Phi(cf) \in L_1 \). Define \( \nu(s) = \mu(\{x: \Phi(cf(x)) \leq s\}) \) and recall that

\[
\int_0^1 u(\Phi(cf)) \, d\mu = \int_0^\infty u(s) \, d\nu(s)
\]

whenever \( u \) is integrable with respect to \( d\nu \). The \( sd\nu(s) \) measure of \([0, \infty)\) is finite (let \( u(s) = s \)) so there exists a function \( \omega: [0, \infty) \rightarrow [0, \infty) \) such that

\[ (a') \quad \int_0^\infty s\omega(s) \, d\nu(s) < \infty, \]

\[ (b') \quad \omega \text{ is nondecreasing,} \]

\[ (c') \quad \omega(0) = 0 \text{ and } \lim_{s \to \infty} \omega(s) = \infty. \]

The function

\[
\Omega_0(s) = \int_0^{\lfloor s \rfloor} \omega(\lfloor s \rfloor) \, ds
\]

is an element of \( \mathcal{C} \) and so is \( \Omega(s) = \Omega_0(\Phi(s)) \) [1, p. 10]. The inequality 1 of §1 gives

\[
\int_0^1 \Omega(cf) \, d\mu = \int_0^\infty \Omega_0(s) \, d\nu(s) \leq \int_0^\infty s\omega(s) \, ds < \infty
\]

from which it follows that \( f \in L^*_\Phi \). To complete the proof, we must show \( \Omega \geq \Phi \). Using inequality 1 again we get

\[
(3) \quad \Omega(s) = \Omega_0(\Phi(s)) \geq [\Phi(s)/2] \omega(\Phi(s)/2);
\]

together with 2 this gives \( \Omega(2s) \geq \Phi(s) \) whenever \( s \geq s_0 \). Here \( s_0 \) is any positive number such that \( \omega(\Phi(s_0)/2) \geq 1 \). This shows that \( \Omega \geq \Phi \).

Let \( \alpha \) be any positive number. There exist positive numbers \( s_0 \) and \( M(\alpha) \) such that \( \Phi(as) \geq M(\alpha)\Phi(s) \) if \( s \geq s_0 \); this is true because \( \Phi \) satisfies \( \ast \). Now this with 3 gives \( \Omega(\alpha s) \geq M(\alpha)\Phi(s)\omega(\Phi(\alpha s)/2)/2 \) for \( s \geq s_0 \) and this in turn gives

\[
\limsup_{s \to \infty} \frac{\Omega(\alpha s)}{\Phi(s)} = \limsup_{s \to \infty} \frac{M(\alpha)\omega(\Phi(\alpha s)/2)}{2} = \infty.
\]

Because \( \alpha \) was arbitrary, we have that \( \Omega \geq \Phi \).

**Lemma.** Suppose \( \Omega \) and \( \Phi \) are two elements of \( \mathcal{C} \) such that for some \( \alpha > 1 \)

\[
\limsup_{s \to \infty} \frac{\Omega(s)}{\Phi(\alpha s)} \geq 1.
\]

Then, if \( t_0 \) is any positive number there exists \( t \geq t_0 \) such that \( \Omega(s) \geq \Phi(s) \) for all \( s \in [t, \alpha t] \).
Proof. Let \( t_0 \) be any positive number and let \( t \geq t_0 \) be any number such that \( \Omega(t) \geq \Phi(\alpha t) \). Let \( l' \) be a straight line through \( (t, \Omega(t)) \) which lies beneath the graph of \( \Omega \); such a line exists because \( \Omega \) is convex (\( l' \) is not necessarily unique). Let \( l \) be the straight line, parallel to \( l' \) which passes through \( (t, \Phi(\alpha t)) \). Let \( u, v \) be the two numbers such that \( u < v \) and \( l \) passes through the points \( (u, \Phi(u)), (v, \Phi(v)) \). By comparing similar triangles we get

\[
\frac{\Phi(v) - \Phi(u)}{v - u} = \frac{\Phi(\alpha t) - \Phi(u)}{t - u}.
\]

This leads directly to the inequality \( v > \alpha t \). For \( s \in [t, \alpha t] \), \( (s, \Phi(s)) \) is beneath the line \( l \) while \( (s, \Omega(s)) \) is above the line \( l' \). Hence \( \Phi(s) \leq \Omega(s) \) for \( s \in [t, \alpha t] \).

Theorem 2. Suppose \( \Phi \in C \) and satisfies **; then \( L_\Phi^* \) is not the union of the Orlicz spaces it contains properly.

Proof. The condition ** implies there exists \( \alpha > 0 \) and \( s_0 \) such that \( s \geq s_0 \) implies \( \Phi(2s) > s\Phi(s) \). Let \( c_n = 2^n s_0 \). Let \( (E_n) \) be a sequence of pairwise disjoint subintervals of \( [0, 1] \) such that \( \Phi(c_n) \mu(E_n) = 2^{-n} \). This is possible because

\[
\Phi(c_n) \leq 2^{n(n-1)/2} s_0^{n-1} \alpha \Phi(s_0).
\]

If we set

\[
f(x) = \sum_{n=1}^{\infty} c_n X_{E_n}(x)
\]

then \( \Phi(f) \in L_1 \) while \( \Phi(2f) \notin L_1 \); in fact

\[
\int_{E_n} \Phi(2f) \, d\mu \geq 2^n s_0 \alpha \Phi(c_n) \mu(E_n) = \alpha s_0.
\]

Suppose \( L_\Phi^* \) is a proper subset of \( L_\Phi^* \). This is the case only if \( \Phi < \Omega \). Let \( k \) be any number in \( (0, 1) \); we will show that \( \Omega(k f) \notin L_1 \) and hence that \( f \notin L_\Omega \). The assertion \( \Omega > \Phi \) implies that

\[
\limsup_{s \to 0} \frac{\Omega(ks)}{\Phi(2s)} = \infty.
\]

Let \( t_0 = c_1 \) and apply the lemma to find \( t_1 \) such that \( s \in [t_1, 2t_1] \) implies \( \Omega(ks) \geq \Phi(2s) \). Having chosen \( t_{n-1} > 2t_{n-1} \) such that \( s \in [t_n, t_{2n}] \) implies \( \Omega(ks) \geq \Phi(2s) \). By induction this yields an infinite sequence of intervals \( [t_n, 2t_n] \) each of which contains one of the numbers \( c_m \).
The proof is completed by observing that
\[ \int_0^1 \Omega(kf) \, d\mu \geq \sum_{i=1}^{\infty} \Phi(2c_{m_i}) \mu(E_{m_i}) \geq \sum_{n=1}^{\infty} \alpha s_0. \]

**Reference**


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**ON A COMBINATORIAL PROBLEM OF ERDÖS**

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Let \( C(n, m) \) denote the binomial coefficient \( \frac{n!}{(m!n-m)!} \). Let \( S \) be a set containing \( N \) elements and let \( X \) be a collection of subsets of \( S \) with the property that if \( A, B \) and \( C \) are distinct elements of \( X \), then \( A \cup B \neq C \). Erdős [1], [2], has conjectured that \( X \) contains at most \( KC(N, \lfloor N/2 \rfloor) \) elements where \( K \) is a constant independent of \( X \) and \( N \). The problem is related to a result of Sperner [3] to the effect that if the collection \( X \) has the more restrictive property that no element of \( X \) contains any other, then \( X \) can have at most \( C(N, \lfloor N/2 \rfloor) \) elements.

We show below that Erdős' conjecture for \( K = 2^{3/2} \) can be deduced directly from Sperner's result.

Let \( L_N \) be defined by

\[
L_N = 2^{\lfloor N/2 \rfloor} C(N - \lfloor N/2 \rfloor, \lfloor N/2 \rfloor) + 2^{N-\lfloor N/2 \rfloor} C(\lfloor N/2 \rfloor, \lfloor N/4 \rfloor).
\]

An easy calculation shows that \( L_N \) is always less than \( 2^{3/2} C(N, \lfloor N/2 \rfloor) \) to which it is asymptotic for large \( N \). We prove:

**Theorem.** If \( X \) is a family of subsets of an \( N \) element set \( S \) such that no three distinct \( A, B, C \) in \( X \) satisfy \( A \cup B = C \), then \( X \) has less than \( L_N \) elements.

**Proof.** For any finite set \( T \) and family \( X \) of subsets of \( T \) define

\[
m_T(X) = \{ A \in X \mid B \in X \text{ and } B \subset A \implies B = A \}.
\]

Received by the editors March 9, 1965.