INCLUSION RELATIONS AMONG ORLICZ SPACES

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This paper contains two results; the first extends to a wide class of Orlicz spaces the statement, due to Krasnosel'skii and Rutickii [1, p. 60], that $L_1$ is the union of the Orlicz spaces which it contains properly; the second shows that for a wide class of spaces this is not true, i.e. there exists a set of Orlicz spaces no one of which is the union of the Orlicz spaces it contains properly. Here the Orlicz spaces are defined on $[0, 1]$ which is given Lebesgue measure $\mu$.

1. We give in this section several definitions together with some elementary results about Orlicz spaces and convex functions.

Let $C$ be the set of convex symmetric functions $\Phi: (-\infty, \infty) \to [0, \infty)$ such that $\Phi(0) = 0$, $\lim_{s \to 0} \Phi(s)/s = 0$ and $\lim_{s \to \infty} \Phi(s) = \infty$. If $\Phi$ and $\Omega$ are two elements of $C$, we say $\Phi \leq \Omega$ if there exist constants $c$ and $s_0$ such that $\Phi(s) \leq \Omega(cs)$ for all $s \geq s_0$. We say $\Phi \sim \Omega$ if $\Phi \leq \Omega$ and $\Omega \leq \Phi$; we say $\Phi \prec \Omega$ if $\Phi \leq \Omega$ but $\Omega \nless \Phi$. If $\Phi_1 \sim \Phi_2$ and $\Phi_1 \leq \Omega_1$ ($\Phi_1 < \Omega_1$) then $\Phi_2 \leq \Omega_2$ ($\Phi_2 < \Omega_2$).

If $\Phi \in C$, then there exists a nondecreasing function $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$, $\lim_{s \to \infty} \phi(s) = \infty$ and

$$\Phi(s) = \int_0^{|s|} \phi(t) \, dt$$

(see [1, p. 5]). This representation for $\Phi$ yields easily the two following inequalities:

1. $$\frac{s}{2} \phi\left(\frac{s}{2}\right) \leq \Phi(s) \leq \phi(s),$$

2. $$2\Phi(s) \leq \Phi(2s).$$

Let

$$L_\phi^\ast = \{f \in L_1: \Phi(cf) \in L_1 \text{ for some positive real number } c\}.$$

The set $L_\phi^\ast$ is called an Orlicz space. It has a unique uniformity which is compatible with the order relation. Since this uniformity does not intervene in what follows, we do not give its definition; for this see [1, p. 69].

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The order relations among the elements of $C$ give rise to order relations among the $L^*_\Phi$ as follows:

(a) $\Phi \leq \Omega$ implies $L^*_\Phi \subset L^*_\Omega$.

(b) $\Phi = \Omega$ implies $L^*_\Phi = L^*_\Omega$.

(c) $\Phi < \Omega$ implies $L^*_\Phi \subset L^*_\Omega$ but $L^*_\Phi \not\subset L^*_\Omega$.

Statements (a) and (b) are direct consequences of the definitions; while (c) is a special case of

(d) $\limsup_{s \to \infty} \frac{\Omega(\alpha s)}{\Phi(s)} = \infty$ for all $\alpha > 0$ implies there exists $f \in L^*_\Phi$ such that $f \notin L^*_\Omega$.

Proof. Let $E_{ij}$ be a pairwise disjoint double sequence of intervals in $[0, 1]$ such that $\mu(E_{ij}) \neq 0$, $i, j = 1, 2, \ldots$. For each pair of natural numbers $(n, i)$ there exists a number $s_{ni} > 0$ such that $s > s_{ni}$ implies $\Phi(s)\mu(E_{ni}) > 2^{-i}n^{-2}$. There exist numbers $c_{ni} > s_{ni}$ such that $\Phi(c_{ni})/n > n^2\Phi(c_{ni})$.

Let $E_{ni}$ be a nonempty subinterval of $E_{ni}$ such that $\Phi(c_{ni})\mu(E_{ni}) = 2^{-i}n^{-2}$ and define

$$f(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}X_{E_{ij}}(x).$$

It is easy to show that $f \in L^*_\Phi$ but $f \notin L^*_\Omega$.

(e) One can use (a), (b) and (d) to show that $\Phi < \Omega$ if $L^*_\Omega$ is a proper subset of $L^*_\Phi$.

2. We say that $\Phi \in C$ satisfies $\ast$ if

$$\limsup_{s \to \infty} \frac{\Phi(2s)}{\Phi(s)} < \infty$$

and we say it satisfies $\ast\ast$ if

$$\liminf_{s \to \infty} \frac{\Phi(2s)}{s\Phi(s)} > 0.$$

These conditions are similar to the $\Delta_2$ and $\Delta_3$ conditioned in [1]. A function $\Phi$ which satisfies $\ast$ grows less rapidly than some power and in addition it grows regularly; while a function $\Phi$ which satisfies $\ast\ast$ grows like exp. It follows that $C$ contains functions which satisfy neither $\ast$ nor $\ast\ast$.

Theorem 1. Suppose $\Phi \in C$ and $\Phi$ satisfies $\ast$; then, $L^*_\Phi$ is the union of the Orlicz spaces it contains properly.

Proof. Let $f \in L^*_\Phi$. We will prove there exists $\Omega$ in $C$ such that $\Phi < \Omega$ and such that $f \in L^*_\Omega$. If $f \in L^*_\infty$, we are finished because $L^*_\infty \subset L^*_\Phi$ properly. We will assume that $f(x) \geq 0$ a.e.; this is in order because
Let \( f \in L^* \) iff \( |f| \in L^* \). Let \( c \) be a positive real number such that \( \Phi(cf) \in L_1 \). Define \( \nu(s) = \mu((x: \Phi(cf(x)) \leq s)) \) and recall that

\[
\int_0^1 u(\Phi(cf)) \, d\mu = \int_0^\infty u(s) \, d\nu(s)
\]

whenever \( u \) is integrable with respect to \( d\nu \). The \( sd\nu(s) \) measure of \([0, \infty)\) is finite (let \( u(s) = s \)) so there exists a function \( \omega: [0, \infty) \to [0, \infty) \) such that

(\(a'\)) \( \int_0^\infty s\omega(s) \, d\nu(s) < \infty \),

(\(b'\)) \( \omega \) is nondecreasing,

(\(c'\)) \( \omega(0) = 0 \) and \( \lim_{s \to \infty} \omega(s) = \infty \).

The function

\[
\Omega_0(s) = \int_0^{|s|} \omega(|s|) \, ds
\]

is an element of \( C \) and so \( \Omega(s) = \Omega_0(\Phi(s)) \) \([1, \text{p. 10}]\). The inequality 1 of \( \S 1 \) gives

\[
\int_0^1 \Omega(cf) \, d\mu = \int_0^\infty \Omega_0(s) \, d\nu(s) \leq \int_0^\infty s\omega(s) \, ds < \infty
\]

from which it follows that \( f \in L^*_\Phi \). To complete the proof, we must show \( \Omega > \Phi \). Using inequality 1 again we get

(3) \( \Omega(s) = \Omega_0(\Phi(s)) \geq [\Phi(s)/2]/\omega(\Phi(s)/2) \);

together with 2 this gives \( \Omega(2s) \geq \Phi(s) \) whenever \( s \geq s_0 \). Here \( s_0 \) is any positive number such that \( \omega(\Phi(s_0)/2) / 2 = 1 \). This shows that \( \Omega \geq \Phi \).

Let \( \alpha \) be any positive number. There exist positive numbers \( s_0 \) and \( M(\alpha) \) such that \( \Phi(\alpha s) \geq M(\alpha)\Phi(s) \) if \( s \geq s_0 \); this is true because \( \Phi \) satisfies \(*\). Now this with 3 gives \( \Omega(\alpha s) \geq M(\alpha)\Phi(s)\omega(\Phi(\alpha s)/2)/2 \) for \( s \geq s_0 \) and this in turn gives

\[
\limsup_{s \to \infty} \frac{\Omega(\alpha s)}{\Phi(s)} = \limsup_{s \to \infty} \frac{M(\alpha)\omega(\Phi(\alpha s)/2)}{2} = \infty.
\]

Because \( \alpha \) was arbitrary, we have that \( \Omega \neq \Phi \).

**Lemma.** Suppose \( \Omega \) and \( \Phi \) are two elements of \( C \) such that for some \( \alpha > 1 \)

\[
\limsup_{s \to \infty} \frac{\Omega(s)}{\Phi(\alpha s)} \geq 1.
\]

Then, if \( t_0 \) is any positive number there exists \( t \geq t_0 \) such that \( \Omega(s) \geq \Phi(s) \) for all \( s \in [t, \alpha t] \).
Proof. Let $t_0$ be any positive number and let $t \geq t_0$ be any number such that $\Omega(t) \geq \Phi(at)$. Let $l'$ be a straight line through $(t, \Omega(t))$ which lies beneath the graph of $\Omega$; such a line exists because $\Omega$ is convex ($l'$ is not necessarily unique). Let $l$ be the straight line, parallel to $l'$ which passes through $(t, \Phi(at))$. Let $u$, $v$ be the two numbers such that $u < v$ and $l$ passes through the points $(u, \Phi(u))$, $(v, \Phi(v))$. By comparing similar triangles we get

$$\frac{\Phi(v) - \Phi(u)}{v - u} = \frac{\Phi(at) - \Phi(u)}{t - u}.$$ 

This leads directly to the inequality $v > at$. For $s \in [t, at]$, $(s, \Phi(s))$ is beneath the line $l$ while $(s, \Omega(s))$ is above the line $l'$. Hence $\Phi(s) \leq \Omega(s)$ for $s \in [t, at]$.

Theorem 2. Suppose $\Phi \in C$ and satisfies $**$; then $L^*_\Phi$ is not the union of the Orlicz spaces it contains properly.

Proof. The condition $**$ implies there exists $\alpha > 0$ and $s_0$ such that $s \geq s_0$ implies $\Phi(2s) > s\Phi(s)$. Let $c_n = 2^ns_0$. Let $(E_n)$ be a sequence of pairwise disjoint subintervals of $[0, 1]$ such that $\Phi(c_n)\mu(E_n) = 2^{-n}$. This is possible because

$$\Phi(c_n) \geq 2^{n(n-1)/2} \frac{n-1}{s_0} \alpha \Phi(s_0).$$

If we set

$$f(x) = \sum_{n=1}^{\infty} c_n X_{E_n}(x)$$

then $\Phi(f) \in L_1$ while $\Phi(2f) \notin L_1$; in fact

$$\int_{E_n} \Phi(2f) \, d\mu \geq 2^ns_0\alpha \Phi(c_n)\mu(E_n) = as_0.$$

Suppose $L^*_\Phi$ is a proper subset of $L^*_\Phi$. This is the case only if $\Phi \leq \Omega$. Let $k$ be any number in $(0, 1)$; we will show that $\Omega(kf) \in L_1$ and hence that $f \notin L_\Omega$. The assertion $\Omega > \Phi$ implies that

$$\limsup_{s \to \infty} \frac{\Omega(ks)}{\Phi(2s)} = \infty.$$

Let $t_0 = c_1$ and apply the lemma to find $t_1$ such that $s \in [t_1, 2t_1]$ implies $\Omega(ks) \geq \Phi(2s)$. Having chosen $t_{n-1}$ choose $t_n > 2t_{n-1}$ such that $s \in [t_n, t_{2n}]$ implies $\Omega(ks) \geq \Phi(2s)$. By induction this yields an infinite sequence of intervals $[t_n, 2t_n]$ each of which contains one of the numbers $c_m$. 

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ON A COMBINATORIAL PROBLEM OF ERDŐS

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Let \( C(n, m) \) denote the binomial coefficient \( n!/(m!n-m!) \). Let \( S \) be a set containing \( N \) elements and let \( X \) be a collection of subsets of \( S \) with the property that if \( A, B \) and \( C \) are distinct elements of \( X \), then \( A \cup B \neq C \). Erdős [1], [2], has conjectured that \( X \) contains at most \( K C(N, [N/2]) \) elements where \( K \) is a constant independent of \( X \) and \( N \). The problem is related to a result of Sperner [3] to the effect that if the collection \( X \) has the more restrictive property that no element of \( X \) contains any other, then \( X \) can have at most \( C(N, [N/2]) \) elements.

We show below that Erdős' conjecture for \( K = 2^{3/2} \) can be deduced directly from Sperner's result.

Let \( L_N \) be defined by

\[
L_N = \sum_{n=1}^{\infty} \Phi(2c_{m_n}) \mu(E_{m_n}) \geq \sum_{n=1}^{\infty} \alpha_s_0.
\]

The proof is completed by observing that

\[
\int_0^1 \Omega(kf) \, df \geq \sum_{n=1}^{\infty} \Phi(2c_{m_n}) \mu(E_{m_n}) \geq \sum_{n=1}^{\infty} \alpha_s_0.
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REFERENCE


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We show below that Erdős' conjecture for \( K = 2^{3/2} \) can be deduced directly from Sperner's result.

Let \( L_N \) be defined by

\[
L_N = 2^{1N/21}C(N - [N/2], [N/4]) + 2^{N-[N/2]}C([N/2], [N/4]).
\]

An easy calculation shows that \( L_N \) is always less than \( 2^{3/2}C(N, [N/2]) \) to which it is asymptotic for large \( N \). We prove:

**Theorem.** If \( X \) is a family of subsets of an \( N \) element set \( S \) such that no three distinct \( A, B, C \) in \( X \) satisfy \( A \cup B = C \), then \( X \) has less than \( L_N \) elements.

**Proof.** For any finite set \( T \) and family \( X \) of subsets of \( T \) define

\[
m_T(X) = \{ A \in X \mid B \in X \text{ and } B \subseteq A \implies B = A \}.
\]

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