SPECTRA OF TENSOR PRODUCTS OF OPERATORS

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1. **Introduction.** Let $\mathcal{H}$ be a Hilbert space, and let A and B denote operators on $\mathcal{H}$. We consider the *tensor product* $A \otimes B$ acting on the product space $\mathcal{H} \otimes \mathcal{H}$. (For a good account of tensor products of Hilbert spaces and operators, the reader may consult [**1**, pp. 22–26].) When $\mathcal{H}$ is finite dimensional, so that A and B can be regarded as matrices, it is a well known and pretty fact that the eigenvalues of $A \otimes B$ are precisely the products of the form $\alpha\beta$ where α, β are eigenvalues of A, B respectively. The purpose of the present note is to prove the following infinite dimensional generalization of this result.

THEOREM. *If A and B are bounded linear operators on an arbitrary Hilbert space $\mathcal{H}$, and if $\sigma(A)$, $\sigma(B)$ denote their respective spectra, then the spectrum of $A \otimes B$ is the set of products $\sigma(A)\sigma(B)$.*

2. In the sequel, all spaces under consideration will be complex Hilbert spaces, and all operators will be assumed to be bounded and linear. The set of all operators on a Hilbert space $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{K})$, and the spectrum of any operator T is denoted, as above, by $\sigma(T)$.

FIRST PROOF OF THE THEOREM. We employ the fact that tensor products of operators on $\mathcal{H}$ can be identified with multiplications on the Hilbert-Schmidt class, regarded as a Hilbert space under the Schmidt norm. The procedure for making this identification is a standard one and the following abbreviated account is lifted from Dixmier [**1**, pp. 93–96]. First, construct the Hilbert space $\mathcal{H}'$ *opposed* to $\mathcal{H}$, i.e., the space whose elements and law of addition are identical with those of $\mathcal{H}$, but with the reversed inner product $(x \mid y) = (y, x)$ and with multiplication of a vector by a scalar redefined as $\lambda \circ x = \bar{\lambda} x$. Note that there is a one-one correspondence $A \leftrightarrow A'$ between $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$ defined by $A'x = Ax$ for all x. Concerning this correspondence, we record the following fact.

LEMMA 1. $\sigma(A') = [\sigma(A)]^*$, *the set of complex conjugates.*

The proof of this lemma consists of an easy computation which we omit.

Presented to the Society, August 25, 1964 under the title *Tensor products of completely locally m-convex algebras* and November 25, 1964 under the title *Tensor products of complete commutative locally m-convex Q-algebras*; received by the editors June 29, 1964.

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Next, form the tensor product $\mathcal{H}'\otimes\mathcal{H}$ and with each decomposable vector $z\otimes x\in\mathcal{H}'\otimes\mathcal{H}$ associate the operator $T_{z,x}\in\mathcal{L}(\mathcal{H})$ defined by

$$T_{z,x}(y) = (y, z)x, \qquad y \in \mathcal{H}$$

(Observe that in this formula z is regarded as an element of $\mathcal{H}$.) The following lemma is essentially [**1**, Proposition 6, page 96].

LEMMA 2. *The correspondence $z\otimes x\rightarrow T_{z,x}$ possesses a unique extension to a Hilbert space isomorphism ϕ of $\mathcal{H}'\otimes\mathcal{H}$ onto the Hilbert space $\mathfrak{N}$ of Hilbert-Schmidt operators on $\mathcal{H}$. Moreover, if A, $B\in\mathcal{L}(\mathcal{H})$, then $A'\otimes B$ is carried by ϕ onto the operator $M(B, A^*)\in\mathcal{L}(\mathfrak{N})$ defined by $M(B, A^*)X=BXA^*$ for all $X\in\mathfrak{N}$.*

Now let ψ be any isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$. (That such an isomorphism exists is assured by the fact that $\mathcal{H}$ and $\mathcal{H}'$ have the same dimension.) For each $A\in\mathcal{L}(\mathcal{H})$, denote by A_0 the element of $\mathcal{L}(\mathcal{H})$ satisfying the equation $A_0' =\psi A\psi^{-1}$. The tensor product of ψ with the identity mapping on $\mathcal{H}$ is an isomorphism of $\mathcal{H}\otimes\mathcal{H}$ onto $\mathcal{H}'\otimes\mathcal{H}$ that carries each operator $A\otimes B\in\mathcal{L}(\mathcal{H}\otimes\mathcal{H})$ onto the operator $A_0'\otimes B\in\mathcal{L}(\mathcal{H}'\otimes\mathcal{H})$. Thus by Lemma 2 we have

$$\sigma(A \otimes B) = \sigma(A_0' \otimes B) = \sigma(M(B, A_0^*)).$$

On the other hand, by Lemma 1

$$\sigma(A) = \sigma(A_0') = [\sigma(A_0)]^* = \sigma(A_0^*),$$

and consequently $\sigma(A)\sigma(B)=\sigma(B)\sigma(A_0^*)$. Thus the proof of the theorem reduces to showing that for arbitrary operators C and D on $\mathcal{H}$,

$$\text{(I)} \qquad \sigma(M(C, D)) = \sigma(C)\sigma(D).$$

Operators of the form $M(C, D): X\rightarrow CXD$ have been studied by Lumer and Rosenblum [**5**], and in particular they show [**5**, Theorem 10] that the equality (I) is valid when $M(C, D)$ is regarded as an operator acting on the Banach space $\mathcal{L}(\mathcal{H})$. While this is not quite our situation (our $M(C, D)$ acts on the Hilbert space $\mathfrak{N}$), it is not hard to see that the arguments of Lumer and Rosenblum remain valid in this case, except that [**5**, Lemma 2] must be given a new proof to take into account the fact that the identity operator is not in $\mathfrak{N}$. Thus we complete the first proof of the theorem by proving the following lemma.

LEMMA 3. *For $A\in\mathcal{L}(\mathcal{H})$, let L_A, $R_A\in\mathcal{L}(\mathfrak{N})$ be defined by setting for $X\in\mathfrak{N}$, $L_A(X)=AX$ and $R_A(X)=XA$, respectively. Then $\sigma(L_A)=\sigma(R_A)=\sigma(A)$.*

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PROOF. Let $J(X)=X^*$ for $X\in\mathfrak{N}$. Then J is an involuntary isometry on $\mathfrak{N}$ and $R_A=JL_{A^*}J$. Since $\sigma(JUJ)=[\sigma(U)]^*$ for every $U\in\mathfrak{L}(\mathfrak{N})$, it suffices to consider L_σ. Next, since $L_{A-\lambda 1}=L_A-\lambda 1$, it suffices to prove that A is invertible on $\mathfrak{H}$ when and only when L_A is invertible on $\mathfrak{N}$. Moreover, one half of this is trivial: if A is invertible on $\mathfrak{H}$, then $L_{A^{-1}}L_A=L_AL_{A^{-1}}=1_{\mathfrak{N}}$. The proof will be completed by showing that if L_A is invertible, then A is invertible too. Accordingly, let L_A be invertible on $\mathfrak{N}$. For any $x\in\mathfrak{H}$, the operator $T_{x,x}$ belongs to $\mathfrak{N}$, so there exists $N\in\mathfrak{N}$ such that $AN=T_{x,x}$. But then $A(Nx)=\|x\|^2x$, and it follows that A maps $\mathfrak{H}$ onto itself. Hence it remains only to verify that A has trivial null space. But if $Ax=0$, then $L_A(T_{x,x})=T_{x,Ax}=0$ so that $T_{x,x}=0$ and consequently $x=0$.

SECOND PROOF OF THE THEOREM. The operator $A\otimes 1_{\mathfrak{H}}$ on $\mathfrak{H}\otimes\mathfrak{H}$ can be regarded as an infinite diagonal matrix with the operator A in each position on the diagonal [1, p. 25]. It follows easily that $\sigma(A\otimes 1_{\mathfrak{H}})=\sigma(A)$, and similarly that $\sigma(1_{\mathfrak{H}}\otimes B)=\sigma(B)$. Since $A\otimes B=(A\otimes 1)(1\otimes B)$, and $(A\otimes 1)$ commutes $(1\otimes B)$, a standard application of the Gelfand representation theorem shows that

$$\sigma(A\otimes B)\subset\sigma(A\otimes 1)\sigma(1\otimes B)=\sigma(A)\sigma(B).$$

The proof of the reverse inclusion $\sigma(A)\sigma(B)\subset\sigma(A\otimes B)$ we split into cases by making a slightly unusual partition of the spectra of A and B. Write $\pi(A)$ for the approximate point spectrum of A [3, page 51] and write $\phi(A)$ for the balance of the spectrum $\sigma(A)\backslash\pi(A)$. Concerning the set $\phi(A)$ we shall need the facts that (i) $\phi(A)$ is an open set and (ii) if $\lambda\in\phi(A)$, then $\bar{\lambda}$ is an eigenvalue of A^*. That $\phi(A)$ is open can be seen in various ways. On the one hand, it follows from the topological fact that $\pi(A)$ is compact and contains the boundary of $\sigma(A)$ (see, e.g. [4, Theorem 4.11.2]). On the other hand, it can be verified directly by a slight modification of any of the standard proofs that the resolvent set is open (see, e.g. [2, §3]). Fact (ii) is an easy consequence of the closed graph theorem.

Case I. $\alpha\in\pi(A)$, $\beta\in\pi(B)$. There exist sequences $\{x_k\}$ and $\{y_k\}$ of unit vectors in $\mathfrak{H}$ along which $A-\alpha 1$ and $B-\beta 1$ tend to zero. But then

$$\begin{aligned}[A\otimes B-\alpha\beta(1\otimes 1)](x_k\otimes y_k)\\ &=[(A-\alpha 1)\otimes B+\alpha 1\otimes(B-\beta 1)](x_k\otimes y_k)\\ &=(A-\alpha 1)x_k\otimes By_k+\alpha x_k\otimes(B-\beta 1)y_k\end{aligned}$$

which tends to zero as $k\to\infty$, so that $\alpha\beta\in\pi(A\otimes B)$.

Case II. $\alpha\in\phi(A)$, $\beta\in\phi(B)$. We have $\bar{\alpha}\in\pi(A^*)$ and $\bar{\beta}\in\pi(B^*)$,

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whence by Case I, $\bar{\alpha}\bar{\beta}\in\sigma(A^*\otimes B^*)=\sigma[(A\otimes B)^*]$. Thus $\alpha\beta\in\sigma(A\otimes B)$.

Case III. $\alpha\in\pi(A)$, $\beta\in\phi(B)$ or $\alpha\in\phi(A)$, $\beta\in\pi(B)$. We treat the case $\alpha\in\pi(A)$, $\beta\in\phi(B)$; the other case is handled similarly. Suppose first that $\alpha=0$. Let β_0 be any element of $\pi(B)$, and note that from Case I, $\alpha\beta=\alpha\beta_0=0\in\sigma(A\otimes B)$. Thus we may assume that $\alpha\neq 0$, and similarly by taking adjoints that $\beta\neq 0$. We know that $\bar{\alpha}\in\sigma(A^*)$; if $\bar{\alpha}\in\pi(A^*)$, then since $\bar{\beta}\in\pi(B^*)$, the desired result follows by appealing to Case I and taking adjoints. Thus we may even assume that $\bar{\alpha}\in\phi(A^*)$. Now introduce a real parameter t, $1\leqq t<\infty$, and consider pairs of the form $[t\bar{\alpha},\ \beta/t]$. For t sufficiently close to 1, we have $t\bar{\alpha}\in\phi(A^*)$ and $\beta/t\in\phi(B)$ since $\phi(A^*)$ and $\phi(B)$ are open sets. Hence there exists $t_0>1$ such that for $1\leqq t<t_0$, $t\bar{\alpha}\in\phi(A^*)$ and $\beta/t\in\phi(B)$ and such that $t_0\bar{\alpha}\in\pi(A^*)$ or $\beta/t_0\in\pi(B)$. Suppose $t_0\bar{\alpha}\in\pi(A^*)$ but $\beta/t_0\in\phi(B)$. Then $\bar{\beta}/t_0\in\pi(B^*)$, so that from Case I we have $(t_0\bar{\alpha})(\bar{\beta}/t_0)=\bar{\alpha}\bar{\beta}\in\sigma([A\otimes B]^*)$, and hence $\alpha\beta\in\sigma(A\otimes B)$. The case $t_0\bar{\alpha}\in\phi(A^*)$ and $\beta/t_0\in\pi(B)$ is handled similarly with the help of Case I, and the only case remaining to be dealt with is $t_0\bar{\alpha}\in\pi(A^*)$, $\beta/t_0\in\pi(B)$. In this case, let t_n be a sequence of real numbers satisfying $1<t_n<t_0$ and $t_n\rightarrow t_0$. Then for each n, $t_n\bar{\alpha}\in\phi(A^*)$, so that $t_n\alpha\in\pi(A)$, and from Case I, $(t_n\alpha)(\beta/t_0)\in\sigma(A\otimes B)$. Since $(t_n/t_0)\alpha\beta\rightarrow\alpha\beta$ and $\sigma(A\otimes B)$ is closed, $\alpha\beta\in\sigma(A\otimes B)$, and the proof is complete.

3. Concluding remarks.

(1) Two proofs of the above theorem are given because each has its own merits. The first argument shows the intimate connection between the operator $A\otimes B$ and the operator $X\rightarrow AXB$ on the Hilbert-Schmidt class. The second proof sets forth a technique for attacking spectral problems that seems fairly useful. In particular, if $\mathcal{K}$ is taken to be any of the Schatten norm ideals [**6**], the spectra of the operators $X\rightarrow AXB$ and $X\rightarrow AX\pm XB$ on $\mathcal{K}$ can be shown to be $\sigma(A)\sigma(B)$ and $\sigma(A)\pm\sigma(B)$ by arguments very similar to the second argument given above. (For $\mathcal{K}=\mathcal{L}(\mathcal{H})$ much more general results are obtained in [**5**].)

(2) It is not hard to see that the line of argument used in Lemma 3 also shows that the approximate point spectra of A and L_A (in the notation of that lemma) coincide. Indeed, even the continuous and residual spectra of A and L_A coincide.

(3) The authors have recently proved that every operator of the form $A\otimes 1_{\mathcal{H}}$ where A is nonscalar and $\mathcal{H}$ is an infinite dimensional Hilbert space is a commutator. This fact, together with the result of this note, shows that there exist commutators with arbitrarily prescribed spectra.

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KIRZBRAUN'S THEOREM AND KOLMOGOROV'S PRINCIPLE[1]

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Let B be a Banach space. A *distance function* p on B is a non-negative valued function which is continuous, positively homogeneous of degree one and subadditive. If A is a set and if x and y map A into B then we write xpy if $p(x(a)-x(b)) \leqq p(y(a)-y(b))$ for all $a, b \in A$. If A is a k-cell, if B is Euclidean space, if p is the norm and if L is Lebesgue area, then Kolmogorov's Principle, K.P., asserts that $Lx \leqq Ly$ if xpy [**H.M.**]. Lebesgue area is a parametric integral of the type considered by McShane [**M**], for smooth enough maps. In this paper we consider other such integrals, not necessarily symmetric, for which a type of K.P. holds. We conclude with a minor application to a Plateau problem.

The proof of K.P. follows from

Kirzbraun's Theorem. *If $A \subset E^n$ and $t: A \rightarrow E^n$ is Lipschitzian, then there exists an extension T of t, $T: E^n \rightarrow E^n$, and T is Lipschitzian with the same constant as t* [**S**].

The proof of the version of K.P. in which we are interested depends upon an embedding of E^n in m, the space of bounded sequences [**B**],

Received by the editors January 11, 1965.

[1] This research was supported in part by National Science Foundation Grant No. GP634.

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