the above mentioned theorem of Cartan and Carathéodory the proof is finished.

References


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A CHARACTERIZATION OF TAME 2-SPHERES IN $E^3$

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In this note, the tame 2-spheres in $E^3$ are characterized partly in terms of homology and the arcs they contain. In a similar way, the compact 2-manifolds with boundary are characterized. If $K$ is a finite topological 2-complex in $E^3$ and $v$ is a vertex of $K$, then $\text{St } v$ is the star of $v$, $\hat{\text{St }} v$ is the open star of $v$, and $\text{Lk } v = \text{St } v - \hat{\text{St }} v$ is the link of $v$. The trivial 1-dimensional homology group of $K$ will be denoted by $H_1(K) = 0$.

An $n$-manifold with boundary is a separable metric space such that each point has a neighborhood whose closure is topologically equivalent to a closed $n$-cell.

Theorem 1. Let $K$ be a finite topological 2-complex in $E^3$ such that

(i) $K$ is connected,
(ii) $\text{Lk } v$ is connected for each vertex $v$ in $K$,
(iii) $H_1(K) = 0$, and
(iv) $K$ contains only tame arcs.

Then $K$ is either a disk or a 2-sphere.

Proof. Since $K$ contains no wild arcs and $\text{Lk } v$ is connected, each 1-simplex in $K$ lies on exactly one or two 2-simplices in $K$ [2]. Since

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1 These results form a part of the author's doctoral dissertation written at the Virginia Polytechnic Institute in 1964 under the direction of Professor P. H. Doyle.
Lk \( v \) is connected, Lk \( v = G \), a connected graph. Again by (iv), no vertex of \( G \) is of order greater than two, and so \( G \) is a 1-manifold with boundary. Thus \( G \) is either an arc or a simple closed curve. So \( St \ v \) is a 2-manifold with boundary, and hence \( K \) is a 2-manifold with boundary. By (iii), \( K \) is either a disk or a 2-sphere.

**Corollary 2.** If \( K \) satisfies the conditions of the Theorem and if, in addition, no arc in \( K \) separates \( K \), then \( K \) is a 2-sphere.

By the addition of one more condition to Corollary 2, we obtain a characterization of tame 2-spheres in \( E^3 \).

**Theorem 3.** A necessary and sufficient condition that a finite connected topological 2-complex \( K \) in \( E^3 \) is a tame 2-sphere is that \( K \) satisfies the following conditions:

(i) Lk \( v \) is connected for each vertex \( v \) in \( K \),
(ii) \( H_1(K) = 0 \),
(iii) \( K \) contains only tame arcs,
(iv) No arc in \( K \) separates \( K \), and
(v) \( E^3 - K \) is locally simply connected at each point of \( K \).

**Proof.** By Corollary 2, \( K \) is a 2-sphere, and by Bing [1], condition (v) insures that \( K \) is tame. Conversely, it is clear that a tame 2-sphere satisfies the conditions.

If the requirement that \( H_1(K) = 0 \) is omitted in Theorem 1, we obtain the following corollary to the proof of Theorem 1.

**Theorem 4.** A finite topological 2-complex \( K \) in \( E^3 \) is a compact 2-manifold with boundary if and only if \( K \) satisfies the following conditions:

(i) \( K \) is connected,
(ii) Lk \( v \) is connected for each vertex \( v \) in \( K \), and,
(iii) \( K \) contains no wild arcs.

**References**


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