ON THE ASYMPTOTIC GROWTH OF SOLUTIONS 
TO A NONLINEAR EQUATION

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We shall consider the nonlinear integral equation

\( \dot{x}(t) - \dot{x}(0) + \int_0^t a(s)g(x(s)) \, ds = h(t), \quad 0 \leq t < \infty, \)

where \( a(t) \) and \( h(t) \) are continuous for \( t \geq 0 \) and \( g(x) \) is continuous for all \( x \). If \( h(t) \) is absolutely continuous on bounded sets, then (1) is equivalent to the differential equation

\( \dot{x}(t) + a(t)g(x(t)) = h(t), \)

so our results will hold for solutions of (2). We shall be primarily interested in the nonlinear oscillator, i.e. \( a(t) > 0, \, xg(x) > 0 \) for \( x \neq 0 \), but in Theorem III below, sign restrictions are removed.

If we consider \( h(t) \) to be a sample function of a Brownian motion process \( h(t, \omega) \) on a probability space \( \Omega \), we see the motivation for considering (1). Equation (2) then represents a nonlinear oscillation driven by “white noise,” an illegitimate process in the sense that for almost all \( \omega \in \Omega \), \( h(t, \omega) \) is not differentiable.

For an account of the probabilistic aspects of (1) for the case in which \( a(t) \) is constant, see [2]. We shall be concerned with bounds on the asymptotic growth of solutions to (1). Various results for the homogeneous case are found in [4], and [5]. Results for the non-homogeneous case when \( a(t) \) is constant are obtained for (1) in [2] and for (2) in [3].

Note that the usual local existence theorems which hold for (2) may be obtained for (1) by considering the equivalent system of equations

\[
\begin{align*}
\dot{x}(t) &= u(t) + h(t), \\
-\ddot{u}(t) &= a(t)g(x(t))
\end{align*}
\]

as in [2]. Here, of course, \( u(t) \) is defined to be \( \dot{x}(t) - h(t) \).

The proofs and results of Theorems I and II are essentially the same as those of Waltman in [4] for the homogeneous case, and depend on the well-known lemma due to Gronwall [1, p. 37].

**Theorem I.** Suppose \( g(x) \) is a monotone nondecreasing odd function

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which is positive for \( x > 0 \) and continuous for all \( x \). Suppose also that \( a(t) \) is positive and absolutely continuous on bounded sets, and that

\[
\int_0^\infty |a'(s)|/a(s)\,ds < \infty.
\]

Let \( x(t) \) be a solution to (1) defined at \( t = 0 \). Then \( x(t) \) can be extended to \([0, \infty)\), and

\[
x(t) = o(t + \int_0^t |h(s)|\,ds)
\]
as \( t \) goes to infinity.

Proof. Condition (3) implies that \( a(t) \) is bounded below by some positive number. In fact, the following is obvious:

**Lemma I.** With \( a(t) \) as above, \( \sup_{0 \leq t \leq \infty} \left[ a(t)/a(\tau) \right] \) is finite. Further, let \( \{T_i\} \) be a nondecreasing sequence of positive numbers and let

\[
\alpha_i = \sup_{T_i \leq t \leq T_{i+1}} \left[ a(t)/a(\tau) \right].
\]

Then \( \prod_{i=1}^\infty \alpha_i < \infty \).

Let \( \beta = \sup_{t \geq 0} a(t)^{-1} \). The theorem will follow easily from the next lemma.

**Lemma II.** Let \([A, B]\) be an interval on which \( x(t) \) is defined and such that \( \dot{x}(t) - h(t) \) does not change sign on this interval. Let

\[
\alpha_{A, B} = \sup_{A \leq t \leq B} \left[ a(t)/a(\tau) \right],
\]

\[
K_{A, B} = \alpha_{A, B} \exp \left\{ \int_A^B |a'(s)|/a(s)\,ds \right\},
\]

\[
M_{A, B} = \int_A^B |h(s)|\,ds.
\]

Pick \( \mu \geq 0 \) so that

\[
\int_0^\mu g(s)\,ds = (2\alpha_{A, B})^{-1}\beta(\dot{x}(A) - h(A))^2.
\]

Then

\[
|x(B)| \leq K_{A, B}(|x(A)| + \mu) + (2 + K_{A, B})M_{A, B}.
\]

Proof of Lemma II. Assume for definiteness that \( \dot{x}(t) - h(t) \) is nonnegative on \([A, B]\). Let
\[ H(t) = \int_{A}^{t} h(s) \, ds \quad \text{and} \quad y(t) = x(t) - H(t). \]

Then
\[ \dot{y}(t) = -a(t)g(y(t) + H(t)) \leq -a(t)g(y(t) - M_{A,B}). \]

Therefore, since \( \dot{y}(t) \geq 0 \) on \([A, B]\),
\[ (4) \quad \dot{y}(t)^2/2 \leq \dot{y}(A)^2/2 - \int_{A}^{t} a(s)g(y(s) - M_{A,B})\dot{y}(s) \, ds \]
for \( A \leq t \leq B \).

Set \( G(x) = \int_{0}^{x} g(s) \, ds \) and note that \( G(x) > 0 \) for \( x \neq 0 \). Integrating by parts in (4), we have
\[ a(t)G(y(t) - M_{A,B}) \leq \dot{y}(A)^2/2 + a(A)G(y(A) - M_{A,B}) \]
\[ + \int_{A}^{t} \left( \frac{a'(s)}{a(s)} \right) a(s)G(y(s) - M_{A,B}) \, ds \]
for \( A \leq t \leq B \). By Gronwall's inequality,
\[ a(B)G(y(B) - M_{A,B}) \leq \left[ \dot{y}(A)^2/2 + a(A)G(y(A) - M_{A,B}) \right] \cdot \exp \left( \int_{A}^{B} \left| \frac{a'(s)}{a(s)} \right| \, ds \right). \]

Recalling that \( g(x) \) is an odd, nondecreasing function, we see that
\[ \int_{0}^{\lvert y(B) - M_{A,B} \rvert} g(s) \, ds \leq K_{A,B} \left[ \beta(\dot{y}(A)^2/2)\alpha_{A,B}^{-1} + \int_{0}^{\lvert y(A) - M_{A,B} \rvert} g(s) \, ds \right] \]
(5)
\[ \leq K_{A,B} \int_{0}^{\lvert y(A) \rvert + M_{A,B} + \mu} g(s) \, ds \]
where we use the fact that \( \dot{y}(A) = \dot{x}(A) - h(A) \).

Now suppose \( p \) and \( q \) are positive numbers and \( \lambda \) is a number \( \geq 1 \). Suppose also that
\[ \int_{0}^{p} g(s) \, ds \geq \frac{1}{\lambda} \int_{0}^{q} g(s) \, ds. \]

Since \( g(x) \) is nondecreasing and positive for \( x > 0 \), it follows that \( p \geq q/\lambda \). From (5), then, we have
\[ \lvert y(B) \rvert \leq K_{A,B}(\lvert y(A) \rvert + \mu) + (1 + K_{A,B})M_{A,B}. \]
Since \( |x(t)| = |y(t) + H(t)| \leq |y(t)| + M_{A, B} \) and \( x(A) = y(A) \), the conclusion of Lemma II is immediate.

Now suppose there is a \( T \geq 0 \) such that \( x(t) \) can be extended past \( T \) and such that in the largest interval \([T, N), N \leq \infty\), to which \( x(t) \) can be extended, \( \dot{x}(t) - h(t) \) does not change sign. If \( N < \infty \), then Lemma II applied to the closed sub-intervals of \([T, N)\) shows that \( x(t) \) is bounded for \( t < N \). Then (1) implies that \( \dot{x}(t) \) is also bounded for \( t < N \). But this implies that \( x(t) \) can be extended to \([0, N)\), contradicting the definition of \( N \). Thus \( N = \infty \). Theorem I follows in this case by applying Lemma II to finite intervals \([T, t]\) and letting \( t \) go to infinity.

If such a \( T \) does not exist, then there must be an unbounded set of zeros of \( \dot{x}(t) - h(t) \). In this case we define two disjoint sets \( S^1 \) and \( S^2 \), with \([0, \infty) = S^1 \cup S^2\), as follows:

\[
S^1 = \{ t \geq 0 \mid \text{There is a neighborhood } (a, b) \text{ of } t \text{ in which } \dot{x}(-) - h(-) \text{ has only a finite number of zeros}\}.
\]

\[
S^2 = \{ t \geq 0 \mid t \in S^1 \}.
\]

If \( t \in S^2 \), then there is a sequence \( t_i \) approaching \( t \) with \( \dot{x}(t_i) - h(t_i) = 0 \). It follows that \( d(\dot{x}(t) - h(t))/dt = 0 \) and by (1), \( x(t) = 0 \).

For each \( t \in S^1 \), define \( I_t \) to be the intersection of \([0, \infty)\) with the union of all neighborhoods of \( t \) in which \( \dot{x}(-) - h(-) \) has only a finite set of zeros. Suppose first that for some \( t_i \), \( I_{t_i} \), is semi-infinite. Then there must exist an increasing and unbounded sequence \( \{T_i\} \) with \((-1)^i(\dot{x}(t) - h(t)) \geq 0 \) on \([T_i, T_{i+1}]\), \( i = 1, 2, \ldots \).

Let \( \gamma_i = K_T, T_{i+1} \) and note that \( \dot{x}(T_i) = h(T_i) \). Lemma II implies that for \( i = 1, 2, \ldots \)

\[
|\dot{x}(T_{i+1})| \leq \gamma_i |\dot{x}(T_i)| + (2 + \gamma_i)M_{T_i, T_{i+1}}.
\]

Lemma I and condition 3 show that \( L = \prod_{i=1}^{\infty} \gamma_i \) is finite. It is easy to show by induction that for \( n = 2, 3, \ldots \),

\[
|\dot{x}(T_n)| \leq \left( \prod_{i=1}^{n-1} \gamma_i \right) |\dot{x}(T_1)| + 3 \left( \prod_{i=1}^{n-1} \gamma_i \right) \int_{T_1}^{T_n} |h(s)| \, ds
\]

\[
\leq L |\dot{x}(T_1)| + 3L \int_{T_1}^{T_n} |h(s)| \, ds,
\]

and Theorem I follows easily in the same way as when \( \dot{x}(t) - h(t) \) does not change sign for \( t \geq T \).

Suppose, on the other hand, that \( I_t \) is bounded for each \( t \in S^1 \). If, for some \( t \in S^1 \), \( I_t = [0, b) \), \( b < \infty \), then \([0, t]\) contains only a finite

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number of zeros of \( \dot{x}(\cdot) - h(\cdot) \). The argument above shows that

\[
| x(t) | \leq L_1(| x(0) | + | \mu_{0,t} |) + 3L_1 \int_0^t | h(s) | \, ds
\]

where

\[
\int_0^{\alpha_{A,B}} g(s) \, ds = (2\alpha_{A,B})^{-1} \beta (\dot{x}(A) - h(A))^2
\]

\[
L_1 = \left[ \sup_{0 \leq t \leq r < \infty} \frac{a(t)}{a(r)} \right] \exp \left\{ \int_0^\infty \frac{a'(s)}{a(s)} \, ds \right\}
\]

and \( \alpha_{A,B} \) and \( \beta \) are as on p. 41, for any \( B \geq A \geq 0 \).

Finally, suppose that for some \( t \), \( I_t = (a, b) \), \( a \geq 0 \), \( b < \infty \). Then \( a \in S^2 \) and \( x(a) = 0 \). For any \( c \) in \( (a, t) \), \( [c, t] \) contains only a finite number of zeros of \( \dot{x}(\cdot) - h(\cdot) \). Therefore, as before,

\[
| x(t) | \leq L_1(| x(c) | + | \mu_{c,t} |) + 3L_1 \int_c^t | h(s) | \, ds.
\]

Letting \( c \) approach \( a \), we see that

\[
| x(t) | \leq 3L_1 \int_a^t | h(s) | \, ds.
\]

We have shown that there is an \( M \) such that for any \( t \geq 0 \),

\[
| x(t) | \leq M + 3L_1 \int_0^t | h(s) | \, ds,
\]

and this completes the proof of Theorem I.

**Theorem II.** Suppose \( a(t) \) is positive, absolutely continuous on bounded sets, and nondecreasing, and \( g(x) \) satisfies the same conditions as in Theorem I. If \( x(t) \) is a solution to (1) defined at \( t = 0 \), then \( x(t) \) can be extended to \( [0, \infty) \) and there is a constant \( c \) such that

\[
| x(t) | \leq c + 3 \int_0^t | h(s) | \, ds, \quad \text{all } t \geq 0.
\]

**Proof.** Let \([A, B], H(t), M_{A,B} \) and \( y(t) \) be as in Lemma II. Suppose \( \dot{x}(t) - h(t) \geq 0 \) on \([A, B]\). Just as in Lemma II, if \( A \leq t \leq B \),

\[
a(t)G(y(t) - M_{A,B}) \leq a(A)G(y(A) - M_{A,B})
\]

\[
+ \dot{y}(A)^2/2 + \int_A^t \frac{a'(s)}{a(s)} a(s)G(y(s) - M_{A,B}) \, ds.
\]
In this case, Gronwall's inequality implies that

\[ a(B)G(y(B) - M_{A,B}) \leq [a(A)G(y(A) - M_{A,B}) + y(A)^2/2]a(B)/a(A). \]

From this,

\[ |y(B)| \leq |y(A)| + \mu + 2M_{A,B} \]

where

\[ \int_{\mu}^\infty g(s) \, ds = (\dot{y}(A)^2/2)a(A)^{-1}, \quad \mu \geq 0. \]

(The same result holds if \( \dot{x}(t) - h(t) \equiv 0 \) on \([A, B]\).)

Since \( |x(B)| \leq |y(B)| + M_{A,B} \) and \( x(A) = y(A) \),

\[ |x(B)| \leq |x(A)| + \mu + 3M_{A,B}. \]

The rest of the proof of Theorem II proceeds much as that of Theorem I and will be omitted.

We note that in both of these theorems, if \( h(t) \) is constant, then all solutions are bounded. (See [4].) We conjecture that the boundedness of all solutions to \( \dot{x} + a(t)g(x) = 0 \) is sufficient to insure that any solution to (1) is \( O(1 + \int_0^\infty h(s) \, ds) \) as \( t \) goes to infinity.

If \( a(t) \) is not bounded below by a positive number, then the conclusion of Theorem I does not hold. In fact, if \( \int_{-\infty}^\infty a(s) \, ds < \infty \), then there is a solution to \( \dot{x} + a(t)x = 0 \) which is asymptotic to a straight line of positive slope [1, p. 103]. In this case, the following theorem provides a bound in the (very restrictive) case \(|g(x)| \leq k|x|^\alpha\) for some constant \( k \) and \( \alpha \in [0, 1] \). We can remove the restrictions on the signs of \( a(t) \) and \( g(x) \) in this case.

We might comment here on a result in [5] to the effect that if \( \int_{-\infty}^\infty a(t) \, dt < \infty \) and \(|g(x)| \leq k|x|^\alpha\) for some \( k \) and \( \alpha \geq 0 \), then all solutions to \( \dot{x} + a(t)g(x) = 0 \) have bounded derivatives. In fact, this is false unless we restrict \( \alpha \) to \([0, 1]\). For example, with \( a(t) = -2/t^4 \), \( g(x) = x^2 \), the function \( x(t) = t^2 \) is such a solution.

**Theorem III.** Suppose \(|g(x)| \leq k|x|^\alpha\) for some \( k \) and some \( \alpha \in [0, 1] \).

Suppose further that \( a(t) \) is continuous for \( t \geq 0 \) and (*) \( \int_{-\infty}^t a(t) \, dt < \infty \).

Let \( x(t) \) be any solution to (1) defined at \( t = 0 \). Then \( x(t) \) can be extended to \([0, \infty)\), and \( x(t) = O(t + \int_0^t |h(s)| \, ds) \) as \( t \) goes to infinity.

**Proof.** We can write, as far as \( x(t) \) is defined,

\[ x(t) = x(0) + t\dot{x}(0) + \int_0^t h(s) \, ds - \int_0^t (t - s)a(s)g(x(s)) \, ds, \]
and if we let $H(t) = \int_0^t h(s) ds$, then there are numbers $c_1$ and $c_2$ such that

$$| x(t) | + 1 \leq c_1 + c_2 t + |H(t)|$$

(6) $$+ \int_0^t (t - s)k | a(s) | (| x(s) | + 1) ds.$$

**Lemma III.** Suppose that $\phi(t)$ is an integrable function on $[0, T]$, and for $t$ in this interval,

$$\phi(t) \leq A(t) + \int_0^t (t - s) B(s) \phi(s) ds,$$

where $B(t)$ is continuous and nonnegative and $A(t)$ is continuous. Let $z(t)$ be any solution to $\ddot{z} - B(t)z = 0$ which is positive for all $t \geq 0$. Then for

$$t \in [0, T], \quad \phi(t) \leq A(t) + z(t) \int_0^t \int_0^t A(u) B(u) z(u) / z(s)^2 du ds.$$

**Proof of Lemma III.** Let

$$R(t) = \int_0^t (t - s) B(s) \phi(s) ds.$$ 

Then $\ddot{R}(t) = B(t) \phi(t) \leq B(t) A(t) + B(t) R(t)$. Let

$$Q(t) = \int_0^t B(s) z(s) ds + \dot{z}(0).$$

Then we have $d [z(t) \ddot{R}(t) - Q(t) R(t)] / dt \leq B(t) A(t) z(t)$. Integrate, divide by $z(t)$ and multiply by $\exp \{ - \int_0^t Q(s) / z(s) ds \}$, and we obtain

$$\frac{d}{dt} \left( R(t) \exp \{- \int_0^t Q(s) / z(s) ds \} \right) \leq \exp \{- \int_0^t Q(s) / z(s) ds \} \cdot \int_0^t B(s) A(s) z(s) / z(t) ds.$$

The result follows by integrating again and noting that

$$Q(t) = \int_0^t B(s) z(s) ds + \dot{z}(0) = \int_0^t \ddot{z}(s) ds + \dot{z}(0) = \ddot{z}(t).$$

Now we note that condition (*) insures the existence of a solution $z_1(t)$ to $\ddot{z}(t) - k | a(t) | z(t) = 0$, with $\eta \leq z_1(t) \leq 1$ for some $\eta > 0$, all $t \geq 0$.  

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Using Lemma III with (6) gives
\[
|\mathbf{x}(t)| + 1 \leq c_1 + c_2 t + |H(t)| \\
+ z_1(t) \int_0^t \frac{1}{z_1(s)^2} \int_0^s k |a(u)||z_1(u)[c_1 + c_2 u + |H(u)|]| \, du \, ds \\
\leq c_1 + c_2 t + |H(t)| \\
+ (k/\eta^2) \int_0^t (t-s) |a(s)||c_1 + c_2 s + |H(s)|| \, ds.
\]

Clearly \( x(t) \) can be extended to \([0, \infty)\). For \( t > 1 \),
\[
\frac{|x(t)| + 1}{t + t |H(t)|} \\
\leq \frac{c_1}{t} + c_2 + 1 + (k/\eta^2) \left[ \int_0^t c_2 |a(s)| s \, ds + \int_0^t |a(s)||c_1 + 1| \, ds \right]
\]
and the result follows from (*).

**Bibliography**


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