EQUIVALENCE OF CONNECTIVITY MAPS AND PERIPHERALLY CONTINUOUS TRANSFORMATIONS

MELVIN R. HAGAN

In [1] and [2] O. H. Hamilton and J. Stallings have shown that a local connectivity mapping, and hence a connectivity mapping, of a locally peripherally connected polyhedron into a regular Hausdorff space is peripherally continuous. The purpose of this paper is to prove the converse of this theorem.

Some definitions will now be recalled. A mapping \( f : S \to T \) is a connectivity mapping if for every connected set \( A \) in \( S \), the set \( g(A) \) is connected, where \( g : S \to S \times T \) is the graph map of \( f \) defined by \( g(p) = (p, f(p)) \) [1, p. 750]. The mapping \( f \) is a local connectivity mapping if there is an open covering \( \{ U_\alpha \} \) of \( S \) such that \( f|_{U_\alpha} \) is a connectivity mapping for every \( \alpha \) [2, p. 249]. The mapping \( f \) is peripherally continuous if for every point \( p \) in \( S \) and for every pair of open sets \( U \) and \( V \) containing \( p \) and \( f(p) \), respectively, there is an open set \( N \subset U \) and containing \( p \) such that \( f(N) \subset V \), where \( F(N) \) is the boundary of \( N \) [1, p. 751]. A space \( S \) is locally peripherally connected if every point has arbitrarily small neighborhoods with connected boundary [2, p. 252].

In this paper \( S \) will denote a connected, locally connected, locally peripherally connected, unicoherent metric space and \( T \) a space such that \( S \times T \) is completely normal.

The following lemma, proved by Stallings [2, p. 255], is used in the proof of Theorem 1.

**Lemma 1.** If \( f : S \to T \) is peripherally continuous, then for every point \( p \) in \( S \) and every pair of open sets \( U \) and \( V \) containing \( p \) and \( (p, f(p)) \), respectively, there is an open connected set \( N \subset U \) and containing \( p \) such that \( F(N) \) is connected and \( g(F(N)) \subset V \).

**Lemma 2.** Let \( W \) be an open connected subset of \( S \) such that \( F(W) \) is connected. Let \( W_1 \) and \( W_2 \) be open connected sets such that \( W_1 \cap W_2 \neq \emptyset \), \( F(W_1) \) and \( F(W_2) \) are connected, and \( \text{cl}(W_1) \cup \text{cl}(W_2) \subset W \). Then there is a connected open set \( W_3 \) such that (1) \( W_1 \cup W_2 \subset W_3 \subset W \), (2) \( F(W_3) \) is contained in \( F(W_1) \cup F(W_2) \), and (3) \( F(W_3) \) is connected.

**Proof.** The proof is similar to the proof of Lemma 1. Let \( X = W_1 \cup W_2 \). Then \( F(X) \) is connected and separates \( F(W) \) and \( X \). Let

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\( C = F(X) \cup \{ y \in W; F(X) \text{ separates } y \text{ and } F(W) \} \) and \( W_3 = \text{component of int } C \text{ containing } X \). Then by standard theorems concerning unicoherence [3, p. 51], \( F(W_3) \subset F(X) \) and \( F(W_3) \) is connected.

The following theorem is the converse of Hamilton's and Stallings' theorem.

**Theorem 1.** If \( f: S \to T \) is peripherally continuous, then \( f \) is a connectivity map.

**Proof.** Suppose that \( f \) is not a connectivity map and let \( A \) be a connected subset of \( S \) such that \( g(A) = M \cup N \), where \( M \) and \( N \) are separated. Let \( g^{-1}(M) = H \) and \( g^{-1}(N) = K \). Then \( A = H \cup K \), where \( H \cap K = \emptyset \). Since \( A \) is connected \( H \) and \( K \) are not separated and hence one must contain a limit point of the other. Let \( p \) be a point of \( H \) that is a limit point of \( K \). Since \( S \times T \) is completely normal there exist open disjoint sets \( U \) and \( V \) in \( S \times T \) containing \( M \) and \( N \), respectively.

Let \( R \) be an open set containing \( p \) such that \( A \) is not contained entirely in \( R \). By Lemma 1 there is an open connected set \( W \) containing \( p \) and contained in \( R \) such that \( W \) and \( F(W) \) are both connected and \( g(F(W)) \subset U \). Since \( p \) is a limit point of \( K \) there is a point \( q \) of \( K \) in \( W \).

Let \( Q \) be the collection of all open connected sets \( D \) such that \( q \) is in \( D \), \( \text{cl}(D) \subset W \), \( F(D) \) is connected, and \( g(F(D)) \subset V \). The collection \( Q \) is nonempty since \( f \) is peripherally continuous at the point \( q \). Denote by \( Q^* \) the point-set union of all sets in \( Q \). Then \( Q^* \) is an open subset of \( W \). Since the connected set \( A \) intersects both \( Q^* \) and \( S - Q^* \), it follows that \( A \cap F(Q^*) \neq \emptyset \).

Since \( F(Q^*) \cap A \neq \emptyset \), then \( F(Q^*) \) either contains a point of \( H \) or a point of \( K \). Suppose there is a point \( h \) in \( F(Q^*) \cap H \). Then there is an open set \( E \) containing \( h \) but not \( q \) such that \( F(E) \) is connected and \( g(F(E)) \subset U \). Since \( h \) is a limit point of \( Q^* \), \( E \) must intersect some set \( D \) belonging to the collection \( Q \). Now \( E \cap D \) since \( h \) is in \( E - D \) and \( D \cap E \) since \( q \) is in \( D - E \). Thus \( E \) and \( D \) both have points interior and exterior to one another and \( F(D) \) and \( F(E) \) being connected implies \( F(D) \cap F(E) \neq \emptyset \). But this contradicts the fact that \( g(F(D)) \subset V \), \( g(F(E)) \subset U \) and \( U \cap V = \emptyset \). Hence \( F(Q^*) \cap H = \emptyset \) and therefore \( F(Q^*) \cap K \neq \emptyset \).

Let \( k \) be a point of \( F(Q^*) \cap K \). Now \( k \) is not a point of \( F(W) \) since \( g(F(W)) \subset U \) and \( g(k) \) is in \( V \). Thus \( k \) is in \( W \) and there is an open connected set \( W_1 \) containing \( k \) and contained in \( W \) such that \( F(W_1) \) is connected, \( \text{cl}(W_1) \subset W \) and \( g(F(W_1)) \subset V \). Since \( k \) is a limit point of \( Q^* \) there is a set \( W_2 \) in the collection \( Q \) such that \( W_1 \cap W_2 \neq \emptyset \).
Now form the set $W_3$ referred to in Lemma 2. By this lemma the set $W_3$ is open, connected, $F(W_3)$ is connected, $\text{cl}(W_3) \subset W$, and $q$ is in $W_3$. Further, $g(F(W_3)) \subset V$ since $F(W_3) \subset F(W_1) \cup F(W_2)$. Therefore $W_3$ possesses all the requirements to belong to $Q$, but $W_3$ is not in $Q$ since $k$ is in $(W_3 \cap F(Q^*))$. Therefore the assumption that $g(A)$ is not connected leads to a contradiction. Hence $f$ is a connectivity map.

Stallings' theorem, [2, p. 253], and Theorem 1 imply, in particular, that on an $n$-cell, $n \geq 2$, into itself there is no distinction among local connectivity maps, connectivity maps, and peripherally continuous transformations. Thus, the question posed on p. 752 of [1] and question 5, p. 262 of [2] are answered. The following theorem will complete the theory of equivalence of the local connectivity maps and the connectivity maps of an $n$-cell, $n = 1, 2, \ldots$, into itself.

**Theorem 2.** If $f$ is a local connectivity map of the closed unit interval $I$ into itself, then $f$ is a connectivity map.

**Proof.** Since $f$ is a local connectivity map there is an open covering $\{U_\alpha\}$ of $I$ such that $f$ restricted to $U_\alpha$ is a connectivity map for each $\alpha$. Since $I$ is compact the covering $\{U_\alpha\}$ can be reduced to an irreducible number of intervals $I_1, \ldots, I_n$, such that $I_i \cap I_{i+1} \neq \emptyset$, and $f$ is a connectivity map on each $I_i$. Then if $K$ is any connected subset of $I$, $K$ is an interval and $K = (K \cap I_1) \cup \cdots \cup (K \cap I_n)$, where each $K \cap I_i$ is an interval contained in $I_i$. Thus $g(K \cap I_i)$ is connected and since $g(K \cap I_i) \cap g(K \cap I_{i+1}) \neq \emptyset$, $g(K)$ is connected. Therefore $f$ is a connectivity map.

**References**


**Oklahoma State University**