ON HAMILTONIAN CIRCUITS IN FINITE GRAPHS

C. ST. J. A. NASH-WILLIAMS

Let G be a finite graph with n ($\geqq 3$) vertices and no loops or multiple edges. Two vertices are *adjacent* if they are joined by an edge. The degree of a vertex v will be denoted by $d(v)$. A *way* is an alternating sequence of distinct vertices and edges of G in which each pair of successive terms are incident and the first and last terms are vertices. The ith vertex of a way W will be denoted by w_i. A *circuit* is obtained from a way with more than two vertices whose first and last terms are adjacent by adding the edge joining them. The number of edges in a way or circuit is its *length*. A circuit of length n is *Hamiltonian*. Pósa [1] proved the following interesting theorem.

Suppose that G satisfies the following conditions:

(i) *for every positive integer k less than* $\frac{1}{2}(n-1)$, *the number of vertices of degree not exceeding k is less than k*,

(ii) *the number of vertices of degree not exceeding* $\frac{1}{2}(n-1)$ *is less than or equal to* $\frac{1}{2}(n-1)$.

Then G has a Hamiltonian circuit.

(We remark that Condition (ii) is contained in Condition (i) if n is even.)

This note presents a slightly different proof of Pósa's theorem, which avoids the construction of additional graphs.

Suppose that G satisfies (i) and (ii). If a component of G has r vertices, the degrees of these vertices cannot exceed $r-1$ and therefore $r>\frac{1}{2}n$ by (i). Therefore each component of G has more than $\frac{1}{2}n$ vertices and so G must be connected. Let m be the maximum of the lengths of the ways in G. Choose a way W of length m such that $d(w_1)+d(w_{m+1})$ is as large as possible. Let S be the set of all vertices w_i such that w_1 is adjacent to w_{i+1}. We note that $w_{m+1} \notin S$. Since there is no way of length $m+1$ in G, w_1 is not adjacent to any vertex not in W, and hence is adjacent to $d(w_1)$ terms of W. Therefore S has cardinal number $d(w_1)$. Moreover, if $w_i \in S$, then

$$w_i,\ w_{i-1},\ \cdots,\ w_1,\ w_{i+1},\ w_{i+2},\ \cdots,\ w_{m+1} \tag{1}$$

are the vertices of a way of length m, and therefore $d(w_i) \leqq d(w_1)$ by the manner in which W was chosen. Hence the degrees of the $d(w_1)$ elements of S do not exceed $d(w_1)$ and therefore by (i), $d(w_1) \geqq \frac{1}{2}(n-1)$.

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By a similar argument, $d(w_{m+1}) \geqq \frac{1}{2}(n-1)$. Moreover, if $d(w_1)$ and $d(w_{m+1})$ were both $\frac{1}{2}(n-1)$, it would follow, since S has $d(w_1)$ elements with degrees not exceeding $d(w_1)$, that $S \cup \{w_{m+1}\}$ was a set of $\frac{1}{2}(n+1)$ vertices with degrees not exceeding $\frac{1}{2}(n-1)$, which is precluded by (ii). It follows that $d(w_1)+d(w_{m+1}) \geqq n$. Therefore w_{m+1} is nonadjacent to at most $d(w_1)$ vertices, and, since w_{m+1} itself is one of these, the $d(w_1)$ elements of S include a vertex w_i adjacent to w_{m+1}.

For this value of i, let X be the way with vertex sequence (1). We note that $x_1 = w_i$ is adjacent to $x_{m+1} = w_{m+1}$. Then, if m were less than $n-1$, the connectedness of G would imply that some vertex v not in X was adjacent to a term x_j of X, which is impossible since in this event

$$v, x_j, x_{j+1}, \cdots, x_{m+1}, x_1, x_2, \cdots, x_{j-1}$$

would be a way of length $m+1$. Therefore $m = n-1$ and X, together with the edge joining x_1 to x_{m+1}, contributes a Hamiltonian circuit of G.

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Reference

1. L. Pósa, *A theorem concerning Hamiltonian lines*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 225–226.

University of Waterloo, Waterloo, Ontario, Canada and
University of Aberdeen, Scotland