It is possible that these together with the constant unitary matrices generate the whole class of such functions, but we have not been able to prove it.

References


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ON THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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Let $\mathcal{C}$ denote the class of functions $f$ regular and univalent in $E = \{z| |z| < 1\}$, which satisfy $f(0) = 0$ and $f'(0) = 1$ and which are close-to-convex in $E$. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the subfamilies of $\mathcal{C}$, made up of functions which are convex and starlike in $E$, respectively. Recently, Libera [2] has shown that if $f$ is a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$, then the function $F(z) = (2/z)f_0 f(t)dt$ is also a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$. It is the purpose of this paper to investigate the converse question. That is, if $F$ is in $\mathcal{S}^*$, what is the radius of starlikeness of the function $f(z) = [1/2] [zF(z)]'$? Similar questions are answered under the assumption that $F$ is in $\mathcal{K}$ or in $\mathcal{C}$. Robinson [5] has shown that if $F$ is only assumed to be univalent in $E$, then $f$ is starlike for $|z| < .38$. He pointed out that it is probable that $f$ is univalent for $|z| < (1/2)$. We obtain this result under the added assumption that $F$ is a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$.

The method of proof used in Theorem 1 has recently been employed by MacGregor [4].

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Theorem 1. If \( F \) is in \( S^* \), then \( f(z) = \left[ 1/2 \right] [z F(z)]' \) is starlike for \( |z| < 1/2 \). This result is sharp.

Proof. Since \( F \) is in \( S^* \), \( \text{Re}[zF'(z)/F(z)] > 0 \) for \( |z| < 1 \). Thus there exists \( \phi \), regular in \( E \), such that \( |\phi(z)| \leq 1 \) for \( z \) in \( E \) and such that

\[
\frac{zf(z) - \int_0^* f(t) dt}{\int_0^* f(t) dt} = \frac{zF'(z)}{F(z)} = \frac{1 - z\phi(z)}{1 + z\phi(z)}.
\]

Thus

\[
f(z) = \frac{2}{z(1 + z\phi(z))} \int_0^* f(t) dt.
\]

Therefore

\[
\frac{zf'(z)}{f(z)} = \frac{-z\phi(z) - z^2\phi'(z)}{1 + z\phi(z)} + \frac{zf(z) - \int_0^* f(t) dt}{\int_0^* f(t) dt}
\]

(1)

\[
= \frac{1 - 2z\phi(z) - z^2\phi'(z)}{1 + z\phi(z)}.
\]

In order to determine where \( f \) is starlike, we must determine those values of \( z \) for which the real part of the right hand side of (1) is positive. This condition is equivalent to

(2) \( \text{Re}[1 - 2z\phi(z) - z^2\phi'(z)][1 + z\phi(z)] > 0 \).

Condition (2) is equivalent to

(3) \( \text{Re}[z^2\phi'(z)][1 + z\phi(z)] < 1 - 2 |z|^2 |\phi(z)|^2 - \text{Re}[z\phi(z)] \).

Using the well known result

\[
|\phi'(z)| \leq \frac{1}{1 - |z|^2} (1 - |\phi(z)|^2) \quad (|z| < 1)
\]

and using the fact that \( \text{Re} \left[ z\phi(z) \right] \leq |z| |\phi(z)| \), we see that condition (3) will be satisfied if

(4) \[
\frac{|z|^2}{1 - |z|^2} (1 - |\phi(z)|^2)(1 + |z||\phi(z)|) < (1 - 2|z||\phi(z)|)(1 + |z||\phi(z)|).
\]
Condition (4) is equivalent to

\[(5) \quad 2 |z|^2 + 2 |z| |\phi(z)| (1 - |z|^2) - |z|^2 |\phi(z)|^2 < 1.\]

Thus, we need only show that condition (5) holds for all functions \(\phi\), regular in \(E\) and satisfying \(|\phi(z)| \leq 1\) for \(z\) in \(E\), provided \(|z| < 1/2\).

If in (5) we let \(a = |z|\) and \(x = |\phi(z)|\), then it is sufficient to show that for any fixed \(a, 0 \leq a < 1/2\), the function \(p(x) = 2a^2 + 2a(1 - a^2)x - a^2x^2\) is bounded above by one for \(0 \leq x \leq 1\). It is easily seen that \(p'(x) > 0, 0 \leq x \leq 1\), provided that \(a < (\sqrt{5} - 1)/2\) and therefore if \(a < 1/2\). Thus, if \(0 \leq a < 1/2\), the maximum value of \(p(x)\), \(0 \leq x \leq 1\), is given by \(q(a) = 2a + a^2 - 2a^3\). Since \(q'(a) > 0\) for \(0 \leq a < 1/2\), \(q(a) < q(1/2) = 1\) for \(0 \leq a < 1/2\). Condition (2) is thus seen to be satisfied, if \(|z| < 1/2\). Hence \(f\) is starlike for \(|z| < 1/2\).

To see that the result is sharp, let \(F(z) = z/(1-z)^2\) which is in \(S^*\). Then, \(f(z) = z/(1-z)^2\) and \(zf'(z)/f(z) = (1+2z)/(1-z) = 0\) for \(z = -1/2\). Thus, \(f\) is not starlike in any circle \(|z| < r\), if \(r > 1/2\).

**Theorem 2.** If \(F\) is in \(K\), then \(f(z) = [1/2][zf(z)]'\) is univalent in \(E\) and is convex for \(|z| < 1/2\). This result is sharp.

**Proof.** We have \(2f'(z) = 2F'(z) + zf''(z)\). Thus

\[(6) \quad 2 \text{Re} \left[ \frac{f'(z)}{F'(z)} \right] = 2 + \text{Re} \left[ \frac{zf''(z)}{F'(z)} \right].\]

Since \(F\) is in \(K\), the right hand side of (6) is positive in \(E\). Thus, \(f\) is close-to-convex relative to \(F\) and therefore is univalent in \(E\).

To show that \(f\) is convex for \(|z| < 1/2\), we notice that \(zf''(z) = [1/2]\cdot[zF'(z)']'\). Since \(F\) is in \(K\), \(zF'\) is in \(S^*\). Therefore, by Theorem 1, \(zf'\) is starlike for \(|z| < 1/2\) and thus \(f\) is convex for \(|z| < 1/2\).

To see that the result is sharp, let \(F(z) = z/(1-z)^2\) which is in \(K\). Then \(f(z) = (2z - z^2)/2(1-z)^2\) and \(1 + [zf''(z)/F'(z)] = (1 + 2z)/(1-z) = 0\) for \(z = -1/2\). Therefore \(f\) is not convex in any circle \(|z| < r\), if \(r > 1/2\).

**Theorem 3.** If \(F\) is in \(C\), then \(f(z) = 1/2[zF(z)]'\) is close-to-convex for \(|z| < 1/2\). This result is sharp.

**Proof.** Since \(F\) is in \(C\), there exists \(G\) in \(S^*\) such that

\[(7) \quad \text{Re} \left[ \frac{zf'(z)}{G(z)} \right] > 0 \quad (|z| < 1).\]

Let \(g(z) = [1/2][zG(z)]'\), then, by Theorem 1, \(g\) is starlike for \(|z| < 1/2\). To prove the theorem, it is sufficient to show that \(\text{Re} \left[ \frac{zf''(z)}{g(z)} \right] > 0\) for \(|z| < 1/2\). We have
Thus, by (7), we may set

\[
\frac{zf'(z)}{G(z)} = \frac{zf(z) - \int_0^z f(t)dt}{\int_0^z g(t)dt}.
\]

(8)

where \(P\) is regular in \(E\) and satisfies \(P(0) = 1\) and \(\text{Re}(P(z)) > 0\) for \(z\) in \(E\). We thus have

\[
zf'(z) = P(z)g(z) + P'(z) \int_0^z g(t)dt.
\]

(9)

Therefore

\[
\frac{zf'(z)}{g(z)} = P(z) + P'(z) \left[ \int_0^z \frac{g(t)dt}{g(z)} \right].
\]

(10)

Using the known result \([1], [3], [6]\)

\[
\left| P'(z) \right| \leq \frac{2 \text{Re}[P(z)]}{1 - |z|^2} \quad (|z| < 1),
\]

we have from (10)

\[
\text{Re} \left[ \frac{zf'(z)}{g(z)} \right] \geq \text{Re}[P(z)] \left[ 1 - \frac{2}{1 - |z|^2} \left| \int_0^z \frac{g(t)dt}{g(z)} \right| \right].
\]

(11)

Moreover

\[
\frac{zg(z)}{\int_0^z g(t)dt} = \frac{[1/2] \{z[zG(z)]'\}}{[1/2] (zG(z))} = 1 + \frac{zG'(z)}{G(z)}.
\]

Since \(G\) is in \(S^*\), \(\text{Re}[zG'(z)/G(z)] > 0\) for \(z\) in \(E\). Thus \(\text{Re}[zg(z)/(\int_0^z g(t)dt)] > 1\) for \(z\) in \(E\). Hence, there exists \(\phi\), regular in \(E\) and satisfying \(\left| \phi(z) \right| \leq 1\) for \(z\) in \(E\), such that \(zg(z)/(\int_0^z g(t)dt) = \)
Therefore

\[ \left| \int_0^z g(t) dt \right| = \left| \frac{z + z^2 \phi(z)}{2g(z)} \right| \leq \frac{1}{2} \left( |z| + |z|^2 \right). \]

Combining (11) and (12) we have

\[ \text{Re} \left[ \frac{zf'(z)}{g(z)} \right] > \text{Re} \left[ P(z) \right] \left[ 1 - \frac{|z|}{1 - |z|^2} \right] \]

(13)

\[ = \text{Re} \left[ P(z) \right] \left[ \frac{1 - |z|}{1 - |z|^2} \right]. \]

The right hand side of (13) is positive provided \(|z| < 1/2\).

To see that the result is sharp, let \( F(z) = z/(1 - z)^2 \) which is in \( S^* \)
and therefore in \( C \). Then \( f(z) = z/(1 - z)^3 \) and \( f'(z) = (1 + 2z)/(1 - z)^4 \)
= 0 for \( z = -1/2 \). Thus, \( f(z) \) is not univalent and therefore not close-to-convex in \(|z| < r\), if \( r > 1/2\).

An interesting subclass of \( C \) is that class made up of functions \( F \)
which satisfy \( \text{Re} \left[ F'(z) \right] > 0 \) for \( z \) in \( E \) [3]. Theorem 3 can be improved
for this subclass.

**Theorem 4.** Let \( F \) be such that \( \text{Re} \left[ F'(z) \right] > 0 \) for \( z \) in \( E \) and let
\( f(z) = \left[ 1/2 \right] \left[ zF(z) \right]' \), then \( \text{Re} \left[ f'(z) \right] > 0 \) for \(|z| < (\sqrt{5} - 1)/2\). This result
is sharp.

**Proof.** Let \( F'(z) = P(z) \) where \( P(0) = 1 \) and \( \text{Re}(P(z)) > 0 \) for \( z \) in \( E \).
We then have

\[ 2f'(z) = 2F'(z) + zF''(z) = 2P(z) + zP'(z). \]

Using again the fact that \( |P'(z)| \leq 2 \text{Re}(P(z))/[1 - |z|^2] \) for \( z \) in \( E \), we have

\[ 2 \text{Re}(f'(z)) \geq 2 \text{Re}(P(z)) - |z| |P'(z)| \]

(14)

\[ \geq 2 \text{Re}(P(z)) \left[ 1 - \frac{|z|}{1 - |z|^2} \right] \]

\[ = 2 \text{Re}(P(z)) \left[ \frac{1 - |z| - |z|^2}{1 - |z|^2} \right]. \]

The right hand side of (14) is positive provided \(|z| < (\sqrt{5} - 1)/2\).

To see that the result is sharp, let \( F(z) = -z - 2 \log(1 - z) \). Then
\( f(z) = \left[ 1/2 \right] \left[ 2z^2/(1 - z) - 2 \log(1 - z) \right] \) and \( f'(z) = (1 + z - z^2)/(1 - z)^2 \)
= 0 for \( z = (1 - \sqrt{5})/2 \).
References


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