A LAW OF THE ITERATED LOGARITHM
FOR STABLE SUMMANDS

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Let $x_n (n=1, 2, 3, \cdots)$ be mutually independent random variables, identically distributed according to the symmetric stable distribution with exponent $\gamma$ ($0<\gamma \leq 2$), i.e., $E[\exp(itx_n)] = \exp(-|t|\gamma)$. Let $S_n = \sum_{k=1}^{n} x_k$. The classical “law of the iterated logarithm” (for the simplest exposition, see Feller [2, pp. 192–195]; see also [3] and [4]) tells us that for $\gamma=2$

$$P \left( \limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right) = 1.$$  

That is, the variables $(1/\sqrt{n})S_n$ again satisfy $E[\exp(it(1/\sqrt{n})S_n)] = \exp(-|t|^2)$, and to achieve a finite lim sup they must be cut down additionally (and multiplicatively) by the factors $(2 \log \log n)^{-1/2}$. For some reason the obvious corresponding statement for the case $\gamma < 2$ does not seem to have been recorded, and it is the purpose of this note to do so.

For $0<\gamma<2$, the variables $n^{-\gamma} S_n$ again satisfy $E[\exp(itn^{-\gamma}S_n)] = \exp(-|t|^\gamma)$. Since the corresponding distribution function $F(x)$ has tail behavior $F(-x) + 1 - F(x) \sim (\text{const})|x|^{-\gamma}$ as $|x| \to \infty$, instead of exponential decrease as in the $\gamma=2$ case, we can expect the “cut down factors” to appear otherwise than as multipliers.

**Theorem.** For $\gamma < 2$

$$P \left( \limsup_{n \to \infty} \left| n^{-\gamma} S_n \right| (\log \log n)^{-1} \right) = 1.$$  

We sketch the proof. It suffices to show that for fixed $\epsilon > 0$, and for almost every sample point, we have

(1) $|n^{-\gamma} S_n| > (\log n)^{(1+\epsilon)\gamma^{-1}}$ finitely often

and

(2) $|n^{-\gamma} S_n| > (\log n)^{(1-\epsilon)\gamma^{-1}}$ infinitely often.

Now the proof proceeds almost exactly as for the $\gamma=2$ case. Thus, to show (1), let $A_n$ be the event that $|S_n| > n^{-1}(\log n)^{(1+\epsilon)\gamma^{-1}}$. Pick

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\( \beta \geq 1 \), and for \( r = 1, 2, 3, \ldots \), let \( n_r \) denote \( \lceil \beta r \rceil \), the largest integer in \( \beta r \). Let \( B_r \) denote the event that \( |S_n| > (n_r)^{-1} (\log n_r)^{1+\epsilon} \gamma^{-1} \) for some \( n \) with \( n_r \leq n < n_{r+1} \). Then \( \lim \sup_{n \to \infty} A_n \subseteq \lim \sup_{n \to \infty} B_r \); and there exists a constant \( c > 0 \) independent of \( r \) such that for all \( r \) \( P(B_r) \leq c P(C_r) \), where \( C_r \) is the event that
\[
(n_{r+1} - 1)^{-1} |S_{n_{r+1}-1}| > \left( \frac{n_r}{n_{r+1} - 1} \right)^{\gamma^{-1}} (\log n_r)^{(1+\epsilon)\gamma^{-1}}.
\]

Since the distribution for \( (n_{r+1} - 1)^{-1} S_{n_{r+1}-1} \) has tail behavior \( \sim (\text{const}) |x|^{-\gamma} \) (cf. [3, pp. 181-182]), we conclude that for some finite constant \( a > 0 \), \( P(C_r) \leq ar^{-1+\epsilon} \gamma^{-1} \), and \( \sum_r P(B_r) < \infty \). Hence by the Borel Cantelli lemma, \( P(\lim \sup_{n \to \infty} A_n) = P(\lim \sup_{n \to \infty} B_r) = 0 \), and (1) holds.

To prove (2), set \( D_r = S_{n_{r+1}} - S_{n_r} \). These are independent variables, and by the Borel Cantelli lemma again we find that for almost every sample point,
\[
(n_{r+1} - n_r)^{-1} |D_r| \geq (\log n_r)^{(1-\epsilon)\gamma^{-1}}
\]
for infinitely many \( r \). Suppose that (2) does not hold on a set of positive probability. Then for almost every sample point in that set,
\[
|n_{r+1}^{-\gamma^{-1}} S_{n_{r+1}}| \geq \left( 1 - \frac{n_r}{n_{r+1}} \right)^{\gamma^{-1}} |D_r| - \left( \frac{n_r}{n_{r+1}} \right)^{\gamma^{-1}} |S_{n_r}|
\]
(3)
\[
\geq \left( 1 - \frac{n_r}{n_{r+1}} \right)^{\gamma^{-1}} (\log n_r)^{(1-\epsilon)\gamma^{-1}} - \left( \frac{n_r}{n_{r+1}} \right)^{\gamma^{-1}} (\log n_r)^{(1-\epsilon)\gamma^{-1}}
\]
for infinitely many \( r \). But for large \( r \) the last difference in (3) dominates \( (\log n_{r+1})^{(1-\epsilon)\gamma^{-1}} \); so (2) does hold almost everywhere. For further details in this paraphrase of the classical case, we refer the reader to Feller [2], loc. cit.

Remark. By restricting \( n \) to subsequences of the form \( n_k = \lceil \beta k^\delta \rceil \) for fixed \( \beta > 1 \) and \( \delta > 1 \), the proof shows that, with probability 1, every point in the interval \( [1, e^{\gamma^{-1}}] \) is a limit point of the sequence \( \{ n^{-\gamma^{-1}} |S_n| (\log \log n)^{-1}, n = 1, 2, 3, \ldots \} \). Now, at least for \( 1 < \gamma < 2 \), 0 is also a limit point, as one can conclude from the general results of Chung and Fuchs (see [1, Theorem 4]). I do not know about the points in the interval \( (0, 1) \).

Added in proof. V. Strassen has pointed out to us that the above theorem follows simply from a result of A. Khinchine, Mat. Sb. 45 (1938); p. 582. However, the present version of the log log law holds also if the common d. f. \( F \) of the \( x_n \) lies in that part of the domain of
normal attraction of a nonnormal stable d.f. $G_\gamma$ ($0 < \gamma < 2$) subject to conditions of the form

$$F(-x) = \frac{c_1}{x^\gamma} + \frac{d_1}{x^\delta} + r_1(x), \quad 1 - F(x) = \frac{c_2}{x^\gamma} + \frac{d_2}{x^\delta} + r_2(x),$$

where $r_1(x) = O(1/x^\gamma)$ and $\gamma < \delta < 1$ (and $r_1(x) + r_2(x)$ are monotone as $x \to \infty$ if $\gamma < 1$). For under these conditions, H. Cramer has shown (in asymptotic expansions for sums of independent random variables with a limiting stable distribution, Sankhya Ser. A 25 (1963), 12–24) that for the d.f. $F_n$ of $S_n$ (suitably shifted and scaled), $F_n(x) - G_\gamma(x) = O(1/n^\gamma)$ uniformly in $x$. Hence in the above proofs, we may replace tail estimates based on $F_n$ by ones based on $G_\gamma$ with an error of at most $O(1/n^\gamma)$. But on subsequences $n_j \subset [c^n], c > 1$, such errors will not affect the convergence or divergence of our series, and the proofs go through as before.

References