UNITARY MATRICES AS BOUNDARY VALUES OF ANALYTIC FUNCTIONS

O. S. ROTHAUS

Let Z be a matricial variable, and $F(Z)$ an analytic matrix valued function. Under what circumstances is $F(Z)$ unitary whenever Z is unitary? We can obtain a partial answer to this question in case $F(Z)$ is entire.

We consider the function $F(zU)$ of a single complex variable z, where U is unitary. $F(zU)$ is a function of the type already considered by Potapov [2], and we can adapt his results to obtain our conclusions.

A matrix valued function $F(z)=[F_{ij}(z)]$ is simply an n by n square array of functions defined for all z in a domain D, usually the unit disc. We will say that $F(z)$ has a property such as analyticity, boundedness, etc., if the property is true of all the component functions.

Now let $F(z)$ be a matrix valued function, analytic in the open unit disc $|z|<1$ and continuous in the closed unit disc. We will suppose further that $F(z)$ is unitary when $|z|=1$. The collection $\mathcal{S}$ of such functions forms a semigroup under (matrix) multiplication. Our first results will give the structure of this semigroup. It is convenient to set some notation:

(i) $\|F(z)\|$ is the determinant of $F(z)$.

(ii) $F^T(z)$ is the transpose of $F(z)$.

(iii) $F^*(z)$ is the conjugate transpose of $F(z)$.

(iv) $f_\alpha(z)=|\alpha|/\alpha\cdot(\alpha-z)/(1-\bar{\alpha}z)$, $0<|\alpha|<1$, and $f_0(z)=z$, called Blaschke factors.

(v) $F_\alpha(z)$ is the matrix

$$\begin{bmatrix} f_\alpha(z) & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

where I_{n-1} is the $n-1$ dimensional identity matrix. (Note that $F_\alpha(z)$ is unitary for $|z|=1$.)

(vi) U_v will always denote a unitary matrix.

We start with the following simple lemma, a variant of the Schwarz reflection principle.

LEMMA 1. *Let $F(z)$ be as described above. Suppose in addition that $\|F(z)\|$ has no zeroes in $|z|<1$. Then $F(z)$ is a constant (necessarily a unitary matrix).*

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Since $\|F(z)\| \neq 0$ for $|z| \leqq 1$, $F^{-1}(z)$ is analytic in $|z| < 1$ and continuous in $|z| \leqq 1$. On the unit circle $|z| = 1$ we have

$$F(z)F^*(z) = I$$

which we write

$$F(z)F^*(1/\bar{z}) = I$$

or

$$F(z) = (F^*(1/\bar{z}))^{-1} \quad \text{for } |z| = 1.$$

But the expression of the right is analytic for $|z| > 1$, and so gives an analytic continuation of $F(z)$ into the extended plane. Consequently, $F(z)$ is constant.

Returning to the general case, note that $\|F(z)\|$ is analytic in $|z| < 1$, continuous in $|z| \leqq 1$ and of absolute value one on $|z| = 1$. Consequently, $\|F(z)\|$ is a constant (of absolute value one) times a finite product of Blaschke factors,

$$\|F(z)\| = e^{i\lambda} \prod_{v=1}^{k} f_{\alpha_v}(z).$$

We have $\|F(\alpha_1)\| = 0$. Let u be a row vector of length one in the null-space of $F(\alpha_1)$:

$$uF(\alpha_1) = 0.$$

We construct a unitary matrix U_1 whose first row is u, so that $U_1F(\alpha_1)$ has a first row identically zero. Thus $U_1F(z)$ has a first row vanishing at $z = \alpha_1$. Consequently, the function $G(z) = (F_{\alpha_1}(z))^{-1}U_1F(z)$ is regular in $|z| < 1$, and, as is immediately verified, belongs to $\mathcal{S}$. $\|G(z)\|$ has one Blaschke factor less in its expansion; i.e.,

$$\|G(z)\| = e^{i\lambda_1} \prod_{v=2}^{k} f_{\alpha_v}(z).$$

After a finite number of such reductions, we arrive at a function to which the result of Lemma 1 is applicable. We have proved:

THEOREM 1. *The semigroup $\mathcal{S}$ is generated by unitary matrices and matrices $F_\alpha(z)$ $(0 \leqq |\alpha| < 1)$.*

If we had assumed at the start that $F(z)$ was an entire function, then we would have $\|F(z)\| = e^{i\lambda}z^k$. Consequently:

COROLLARY 1. *The sub-semigroup of $\mathcal{S}$ consisting of entire functions is generated by the unitary matrices and the single matrix $F_0(z)$. Further-*

more, if $F(z)$ is an entire function in $\mathcal{S}$, then $\|F(z)\| = e^{i\lambda}z^k$, and the entries in $F(z)$ are all polynomials of degree at most k.

Potapov has many interesting results on the uniform closure of the semigroup $\mathcal{S}$.

We return now to the question in the initial paragraph of this paper. We have that

$$\|F(zU)\| = c(U)z^{K(U)}$$

where, in fact, since the unitary group is connected, $K(U)$ must be constant K. Now from Corollary 1, we know that the entries in $F(zU)$ are all polynomials of degree at most K. Hence in the expansion of $F(Z)$ in a power series, the homogeneous terms of weight greater than K must vanish on the unitary group. This is known to insure their vanishing identically. Thus we have:

THEOREM 2. *Under the hypothesis described above, the entries in $F(Z)$ are polynomials in the entries of Z.*

Somewhat more can be proved easily. Our argument above will show that $\|F(Z)\|$ is a homogeneous polynomial of weight K. For Z unitary we have:

$$F(Z)F^*(Z) = I$$

which we write:

$$F(Z)F^*(Z^{*-1}) = I.$$

Since $F^*(Z^{*-1})$ is analytic where it is defined, the above equation is true for all Z. Put $\operatorname{ad} Z = \|Z\| Z^{-1}$. $\operatorname{ad} Z$ is a polynomial in Z. By taking determinants above we obtain:

$$\|F(Z)\| \left\| F^*\left(\frac{1}{\|Z^*\|} \operatorname{ad} Z^*\right)\right\| = I.$$

Using the homogeneity of $\|F(Z)\|$ we have:

$$\|F(Z)\| \, \|F^*(\operatorname{ad} Z^*)\| = \|Z\|^K.$$

Since $\|Z\|$ is irreducible, $\|F(Z)\| = c\|Z\|^L$.

In the special case that the dimension of Z is that of $F(Z)$, we note the following entire functions which are unitary when Z is unitary:

(i) Z,
(ii) Z^T,
(iii) $\operatorname{ad} Z$,
(iv) $\operatorname{ad} Z^T$,

(v) $$\begin{pmatrix} \|Z\| & 0 \\ 0 & I \end{pmatrix}.$$

It is possible that these together with the constant unitary matrices generate the whole class of such functions, but we have not been able to prove it.

References

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2. V. P. Potapov, *The multiplicative structure of J-contractive matrix functions*, Amer. Math. Soc. Transl. (2) 15 (1960), 131–243.

Institute for Defense Analyses, Princeton, New Jersey

ON THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

A. E. LIVINGSTON

Let $\mathcal{C}$ denote the class of functions f regular and univalent in $E=\{z \mid |z|<1\}$, which satisfy $f(0)=0$ and $f'(0)=1$ and which are close-to-convex in E. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the subfamilies of $\mathcal{C}$, made up of functions which are convex and starlike in E, respectively. Recently, Libera [2] has shown that if f is a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$, then the function $F(z)=(2/z)\int_0^z f(t)dt$ is also a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$. It is the purpose of this paper to investigate the converse question. That is, if F is in $\mathcal{S}^*$, what is the radius of starlikeness of the function $f(z)=[1/2][zF(z)]'$? Similar questions are answered under the assumption that F is in $\mathcal{K}$ or in $\mathcal{C}$. Robinson [5] has shown that if F is only assumed to be univalent in E, then f is starlike for $|z|<.38$. He pointed out that it is probable that f is univalent for $|z|<(1/2)$. We obtain this result under the added assumption that F is a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$.

The method of proof used in Theorem 1 has recently been employed by MacGregor [4].

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