1. Introduction. In considering the stability (i.e., the asymptotic smoothness of higher order iterates) of continuous smoothing problems in least square approximation, H. S. Wilf [7(a)] introduces a certain inequality ((2) below) involving Bessel functions.

His argument in support of this inequality requires correction [7(b)], which it is our purpose here to supply.

2. The inequality. Defining

\[ h_{\nu \lambda} (\theta) = 1 - \frac{\int_0^{\theta} t^{-\lambda} J_{\nu}(t) dt}{\int_0^{\infty} t^{-\lambda} J_{\nu}(t) dt}, \]

where \( J_{\nu}(t) \) is the Bessel function of first kind and order \( \nu \), the inequality in question is

\[ -1 < h_{\nu \lambda} (\theta) < 1 \quad (\theta \neq 0), \]

for \( \lambda = 1/2, \nu = 2k + 3/2, k \) a sufficiently large positive integer.

3. Preliminaries. In verifying (2) for appropriate \( \lambda \) and \( \nu \), some preliminary results will be needed. The first is a corrected version of Wilf’s formula (8), which we establish in a somewhat extended form:

\[ \lim_{\nu \to \infty} \frac{\nu^\lambda}{\nu^\lambda} \int_0^{j_{\nu, 1}} t^{-\lambda} J_{\nu}(t) dt = \frac{1}{3} + \frac{1}{3} \int_0^c [J_{1/3}(t) + J_{-1/3}(t)] dt = 1.2743521, \]

where \( \lambda > -1/2, j_{\nu, 1} \) is the first positive zero of \( J_{\nu}(t) \), and \( c \) is the least positive zero of \( J_{1/3}(t) + J_{-1/3}(t) \).

Proof of (3). The denominator of the first member of (3) is equal to \( (\nu/2)^{\lambda} \Gamma [(\nu + 1 - \lambda)/2]/\Gamma [(\nu + 1 + \lambda)/2] \) (cf., e.g., [4, p. 414 (11)]),

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and so has limit equal to 1. The limit of the numerator is equal to the subsequent members of (3) by \([4, p. 409 (4)]\).\(^2\)

Another result to be used is

\[
\int_0^{j_{\nu_2}} t^{-\lambda} J_\nu(t) dt > 0 \quad \text{with} \quad \lambda < \nu + 1 \quad \text{(to insure convergence of the integral at the origin)}
\]

where \(j_{\nu_2}\) is the second positive zero of \(J_\nu(t)\).

**Proof of (4)(i).** For small positive \(\epsilon\), the second mean-value theorem applies so that

\[
\int_0^{j_{\nu_2}} t^{-\lambda} J_\nu(t) dt > \int_\epsilon^{j_{\nu_2}} t^{-\lambda} J_\nu(t) dt,
\]

\[
= \epsilon^{-\lambda} \int_\epsilon^{\eta} J_\nu(t) dt, \quad \epsilon < \eta < j_{\nu_2}.
\]

If \(\eta \leq j_{\nu_1}\), then this last integral is positive for \(\nu > -1\), and (4)(i) is proved. If \(\eta > j_{\nu_1}\), then this last integral clearly exceeds

\[
\epsilon^{-\lambda} \int_\epsilon^{j_{\nu_2}} J_\nu(t) dt
\]

and this, in turn, is positive for sufficiently small \(\epsilon > 0\), in view of R. G. Cooke's result [3] that

\[
\int_0^{j_{\nu_2}} J_\nu(t) dt > 0, \quad \nu > -1.
\]

**Proof of (4)(ii).** The same argument applies here to

\[
\int_0^{j_{\nu_2}} t^{-\lambda} [t^{1/2} J_\nu(t)] dt, \quad \lambda > -1/2, \quad \nu > 1/2,
\]

in view of E. Makai's result [6] that

\[
\int_0^{j_{\nu_2}} t^{1/2} J_\nu(t) dt > 0, \quad \nu > 1/2.
\]

\(^2\) The results of [4] are summarized and extended in [5].

\(^3\) Z. Ciesielski has mentioned that (4)(i) and (ii) can be inferred from the Cooke and Makai results, respectively, also via Theorem 1a of [1].

\(^4\) Here the Bessel function of the first kind, \(J_\nu(t)\), can be replaced by an *arbitrary* solution of the Bessel equation, say \(C_\nu(t)\), normalized so as to be positive for \(t\) between zero and the first positive zero, with the parameters \(\lambda, \nu\) restricted so as to insure convergence of the integral. This extension arises because [6], used in the proof of (4) (ii), covers this case.
A simplified version (cf. [2]) of these proofs (the introduction of \( \varepsilon \) being superfluous) shows that

\[
(-1)^p \int_{j_{\nu, p+2}}^{j_{\nu, p+2}} t^{-\lambda} J_\nu(t) dt > 0 \quad \begin{cases} 
(\text{i}) & \lambda \geq 0, \quad \nu > -1 \\
(\text{ii}) & \lambda > -1/2, \quad \nu > 1/2,
\end{cases}
\]

where \( j_{\nu, p} \) is the \( p \)th positive zero of \( J_\nu(t) \), \( p = 1, 2, \ldots \).

4. Proof of the inequality. That \( h_{\nu, \lambda}(\theta) < 1 \) for \( \lambda < \nu + 1 \), and either \( \lambda \geq 0, \nu > -1 \) or \( \lambda > -1/2, \nu > 1/2 \) follows at once by combining (4) and (5), since they imply the positivity of the numerator in (1), for all \( \theta \neq 0 \). The denominator is also positive (its value is contained in the proof of (3)).

To show that \( h_{\nu, \lambda}(\theta) > -1 \) for appropriate \( \nu, \lambda \), it suffices to show, as Wilf points out [7(a), p. 937], that the ratio of the integrals in (1) is less than 2. From (4) and (5) it is clear that the maximum of this ratio is achieved for \( \theta = j_{\nu, 1} \). But, for \( \lambda > -1/2 \), and all sufficiently large \( \nu \), this ratio must be less than 2, since the constant term in (3) is 1.2743521 < 2.

Thus, (2) is established for all sufficiently large \( \nu \), if \( \lambda > -1/2 \) and \( \lambda < \nu + 1 \). In particular, (2) holds for \( \lambda = 1/2, \nu = 2k + 3/2 \), for all sufficiently large positive integers \( k \), the case relevant to [7(a)].

References


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