Let $G$ be an arbitrary discrete group and let $\Gamma = C[G]$ be its group algebra over the complex numbers $C$. If $\mathcal{R}$ is an irreducible representation of the algebra then $\mathcal{R}(\Gamma) = P$ is primitive and hence isomorphic to a dense set of linear transformations over $D$, the commuting ring of $\mathcal{R}$ [4, p. 28]. Let $E$ be the center of $D$. If $\dim_EL < \infty$ then we say that $\mathcal{R}$ is finite and since $P$ is central simple over $E$ [4, p. 122] we have $\dim_EL = m^2$. We set $m = \deg \mathcal{R}$, the degree of $\mathcal{R}$. If $G$ is finite then $C$ is always the commuting ring of $\mathcal{R}$ so this agrees with the usual definition of degree.

Again let $P = \mathcal{R}(\Gamma)$. Then by a theorem of Amitsur [1] $\deg \mathcal{R} \leq n$ if and only if for every $2n$ elements $x_1, \ldots, x_{2n}$ in $P$ we have

$$[x_1, \ldots, x_{2n}] = \sum \pm x_{i_1}x_{i_2}\cdots x_{i_{2n}} = 0.$$ 

The above is known as the standard identity of degree $2n$. For infinite discrete groups, representation theory is not particularly well behaved. Therefore we will make use of these identities in $C[G]$.

If $g \in G$ we say that $g$ is in the kernel of $\mathcal{R}$ if and only if $\mathcal{R}(g) = \mathcal{R}(1) = 1$. We set $\kappa_n(G) = \bigcap \ker \mathcal{R}$ where $\mathcal{R}$ runs over all irreducible representations of degree $> n$. We study groups $G$ with $\kappa_n(G) > 1$. It is convenient to let $b(G) = \text{lub}$ of the degrees of the irreducible representations of $G$. If $b(G) \leq n$ then trivially $\kappa_n(G) = G$. Thus we will be interested mainly in groups with $b(G) > n$.

**Theorem 1.** Let $I_n = I_n[G]$ be the linear subspace of $C[G]$ spanned by all terms of the form $[x_1, \ldots, x_{2n}]$ with $x_i \in C[G]$. Then $g \in \kappa_n(G)$ if and only if $(1 - g)I_n = 0$.

**Proof.** First let $g \in \kappa_n(G)$. Let $\mathcal{R}$ be any irreducible representation of $C[G]$ and consider $\mathcal{R}((1 - g)I_n)$. If $\deg \mathcal{R} > n$ then $\mathcal{R}(1 - g) = 0$. If $\deg \mathcal{R} \leq n$ then $\mathcal{R}(I_n) = 0$. Hence in either case $\mathcal{R}((1 - g)I_n) = 0$. Since this holds for all $\mathcal{R}$ and $C[G]$ is semi simple [5, Theorem 5.2] this yields $(1 - g)I_n = 0$.

Conversely let $(1 - g)I_n = 0$. Let $J_n = \{a \in C[G] | aI_n = 0\}$ so that $J_n$ is clearly a left ideal of $C[G]$. To show that it is a right ideal we need only show for $h \in G$ that $J_nh \subseteq J_n$. Since clearly $h^{-1}I_nh = I_n$ we have

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\[(J_n h) I_n = (J_n h) (h^{-1} I_n h) = J_n I_n h = 0\]

and so \(J_n\) is a two-sided ideal. Let \(\mathfrak{R}\) be an irreducible representation of \(C[G]\) of degree \(> n\). Let \(V\) be the corresponding left \(C[G]\)-module and set \(V^* = \{ v \in V \mid J_nv = 0 \}\). Since \(J_n\) is a right ideal, \(V^*\) is a submodule of \(V\). Hence either \(V^* = V\) or \(V^* = 0\). Now \(\deg \mathfrak{R} > n\) so \(\mathfrak{R}(C[G])\) does not satisfy the standard identity of degree \(2n\). Hence \(\mathfrak{R}(I_n) \neq 0\) and \(I_n V \neq 0\). But \(J_n(I_n V) = 0\) so \(I_n V \subseteq V^*\) and hence \(V^* = V\). Since \((1-g) \in J_n\) and \(J_n V = 0\) we have \(\mathfrak{R}(g) = \mathfrak{R}(1)\) and the result follows.

Now \(I_n[G]\) is spanned as a linear space over \(C\) by all terms of the form \([g_1, \cdots, g_{2n}]\) with \(g_i \in G\). Hence we have the following.

**Corollary 2.** Let \(b(G) > n\). Then \(| \mathfrak{R}_n(G) | \leq \frac{1}{r} (2n)!\).

**Proof.** Now \(b(G) > n\) implies that \(I_n[G] \neq 0\). Thus we can find \(g_i \in G\) with \(\beta = [g_1, \cdots, g_{2n}] \neq 0\). If \(g \in \mathfrak{R}_n(G)\) then \((1-g)\beta = 0\) so that \(g \beta = \beta\). If \(\beta = \sum c_i h_i\) with \(c_i \in C\) and \(h_i \in G\) then \(\mathfrak{R}_n(G)\) permutes all those \(h_i\) with \(c_i > 0\). Thus \(\mathfrak{R}_n(G)\) divides \(r\), the number of such \(h_i\). Since \(\beta \neq 0\) and \(\sum c_i = 0\) we see that \(1 \leq r \leq \frac{1}{r} (2n)!\) and the result follows.

If \(H\) is a subgroup of \(G\) then \(C[H]\) is naturally embedded in \(C[G]\). Moreover in this embedding \(I_n[H] \subseteq I_n[G]\). With this remark we have

**Corollary 3.** Let \(H\) be a subgroup of \(G\) with \(b(H) > n\). Then \(\mathfrak{R}_n(G) \subseteq \mathfrak{R}_n(H)\).

**Proof.** Since \(b(H) > n\) there exists \(h_i \in H\) with \(\beta = [h_1, \cdots, h_{2n}] \neq 0\). Let \(g \in \mathfrak{R}_n(G)\). Since \(\beta = g \beta\) we see clearly that \(g \in \langle h_1, \cdots, h_{2n} \rangle \subseteq H\). Now \((1-g) I_n[G] = 0\) implies \((1-g) I_n[H] = 0\) so \(g \in \mathfrak{R}_n(H)\).

The following result essentially reduces the study of groups \(G\) with \(b(G) > n\) and \(\mathfrak{R}_n(G) > 1\) to a study of finite groups.

**Theorem 4.** Let \(b(G) > n\) and \(\mathfrak{R}_n(G) > 1\). Let \(\beta_1, \cdots, \beta_m\) be a finite number of nonzero elements of \(C[G]\). Then we can find subgroups \(H\) and \(N\) of \(G\) such that

(i) \(\beta_1, \cdots, \beta_m \in C[H]\);
(ii) \(b(H) > n\);
(iii) \(\mathfrak{R}_n(H) = \mathfrak{R}_n(G)\);
(iv) \(N\) is a normal subgroup of \(H\) with \(\overline{H} = H/N\) finite and \(b(\overline{H}) > n\);
(v) under the natural homomorphism \(C[H] \to C[\overline{H}]\) we have \(\beta_i \to \overline{\beta_i} \neq 0\) and \(\mathfrak{R}_n(G) \cong [\mathfrak{R}_n(G)]^\sim = \mathfrak{R}_n(\overline{H})\).

**Proof.** First we show that we can find group elements \(g_{ij}\) with
$i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, 2n$ such that $g \in \mathfrak{K}_n(G)$ if and only if for all $i$

$$(1 - g)[g_{i,1}, g_{i,2}, \cdots, g_{i,2n}] = 0.$$ 

Let $\mathcal{S}$ be the set of all terms $\alpha = [g_1, \cdots, g_{2n}] \neq 0$ with $g_i \in G$. For each such $\alpha \in \mathcal{S}$ set $\mathfrak{K}^\alpha = \{g \in G \mid (1 - g)\alpha = 0\}$. As in the proof of Corollary 2, $|\mathfrak{K}^\alpha| \leq \frac{1}{2}(2n)!$. By Theorem 1, $\mathfrak{K}_n(G) = \bigcap \mathfrak{K}^\alpha$. Since each $\mathfrak{K}^\alpha$ is finite, clearly only a finite intersection is required.

For the remainder of the proof fix such a set $\{g_{ij}\} \subseteq G$. Let $H$ be a finitely generated subgroup of $G$ with $H \supseteq \{g_{ij}\}$ and $\beta_1, \cdots, \beta_m \in \mathcal{C}[H]$. Such groups clearly exist. Clearly $b(H) > n$ and we have $\mathfrak{K}_n(H) \supseteq \mathfrak{K}_n(G)$. But if $h \in \mathfrak{K}_n(H)$ then $(1 - h)[g_{i,1}, \cdots, g_{i,2n}] = 0$ for all $i$ so $h \in \mathfrak{K}_n(G)$. Hence $\mathfrak{K}_n(H) = \mathfrak{K}_n(G)$. With this choice of $H$ we have (i), (ii) and (iii) of Theorem 4 satisfied.

We now show that $H$ is a subdirect product of finite groups. Fix $g \in \mathfrak{K}_n(H)$ with $g \neq 1$. Let $h$ be any nonidentity element of $H$. In $\mathcal{C}[H]$ the expression $\gamma = (1 - g)(1 - h)$ is nonzero since otherwise $1 + gh = g + h$ and so $1 = g$ or $h$. Since $\mathcal{C}[H]$ is semisimple there exists an irreducible representation $\mathcal{R}$ of $\mathcal{C}[H]$ with $\mathcal{R}(\gamma) \neq 0$. Hence $\mathcal{R}(g) \neq 1$ and $\mathcal{R}(h) \neq 1$. The first of these implies that $\deg \mathcal{R} \leq n$. Hence we conclude that $H$ is a subdirect product of linear groups of finite degree. By Proposition 7.3 of [3], each such linear group being finitely generated is the subdirect product of finite groups. Hence the result follows.

Now only a finite number of group elements of $H$ occur in the expressions for the $\beta_i$ and the $[g_{i,1}, \cdots, g_{i,2n}]$. Let these be $h_1, \cdots, h_s$. Then we can write $\beta_i = \sum c_{ij} h_j$ and $[g_{i,1}, \cdots, g_{i,2n}] = \sum d_{ij} h_j$ with $c_{ij}, d_{ij} \in \mathcal{C}$. Let $\mathfrak{D}$ be the finite set containing (1) $\mathfrak{K}_n(H)$, (2) all elements of the form $h_j h_k^{-1}$, and (3) all elements of the form $h_j h_j^{-1} h_k h_k^{-1}$. By the above there exists a normal subgroup $N$ of $H$ of finite index with $N \cap \mathfrak{D} = \{1\}$. We show now that with this $N$, (iv) and (v) of Theorem 4 follow.

Since $N \cap \mathfrak{D} = \{1\}$ and $h_j h_k^{-1} \in \mathfrak{D}$ it follows that under the homomorphism $H \to \overline{H} = H/N$, that $\overline{h}_j$ (the image of $h_j$) is not equal to $\overline{h}_k$. With this we see that $\overline{\beta}_i \neq 0$ and that $[\overline{g}_{i,1}, \cdots, \overline{g}_{i,2n}] \neq 0$. The latter implies in addition that $b(\overline{H}) > n$. Since any irreducible representation of $\mathcal{C}[\overline{H}]$ can be viewed as one of $\mathcal{C}[H]$ we have $N \mathfrak{K}_n(H)/N \subseteq \mathfrak{K}_n(H)$. But $N \mathfrak{K}_n(H)/N \approx \mathfrak{K}_n(\overline{H})/(\mathcal{N} \cap \mathfrak{K}_n(H)) \approx \mathfrak{K}_n(H)$ since $N \cap \mathfrak{K}_n(H) = 1$. Hence $\mathfrak{K}_n(H) = \mathfrak{K}_n(G)$ is contained isomorphically in $\mathfrak{K}_n(\overline{H})$. We need only show that the isomorphism is onto. Let $\overline{g} \in \mathfrak{K}_n(\overline{H})$ with $g$ an inverse image of $\overline{g}$. It suffices to show that $g \in N \mathfrak{K}_n(H)$.
Since \( \tilde{g} \in \mathfrak{g}_n(H) \) we have for all \( i \)
\[
\tilde{g} [\tilde{g}_{i1}, \ldots, \tilde{g}_{i2n}] = [\tilde{g}_{i1}, \ldots, \tilde{g}_{i2n}].
\]
In \( C[G] \) this yields clearly
\[
g(\sum d_{ij} h_j) = \sum d_{ij} n_{ij} h_j
\]
with \( n_{ij} \in N \). This follows since all the \( h_j \) are distinct. We show now that all the \( n_{ij} \) are equal. Consider one such element \( n_{ij} h_j \). This comes from a term \( g h_j' \) on the left of the above equation. Thus \( g h_j' = n_{ij} h_j \) or \( g = n_{ij} h_j h_j^{-1} \). Replacing \( i \) by \( i' \), \( j \) by \( k \), and \( j' \) by \( k' \) we also have \( g = n_{i'k} h_k h_k^{-1} \). Thus
\[
n_{ij}^{-1} n_{i'k} = h_j h_j^{-1} h_k h_k^{-1} \in N \cap 3 = \{1\}
\]
so \( n_{ij} = n_{i'k} \). Let their common value be \( n_{11} \). Then for all \( i \)
\[
n_{11}^{-1} g [g_{i1}, \ldots, g_{i2n}] = [g_{i1}, \ldots, g_{i2n}].
\]
By the choice of the \( g_{ij} \) this implies that \( n_{11}^{-1} g \in \mathfrak{g}_n(G) = \mathfrak{g}_n(H) \) and \( g \in N \mathfrak{g}_n(H) \). This completes the proof.

As an application of the above result we prove

**Theorem 5.** Let \( \mathfrak{g}_n(G) > 1 \). Then \( b(G) = n^2 \).

**Proof.** Suppose by way of contradiction that \( b(G) > n^2 = m \). Then we can find group elements \( g_1, \ldots, g_{2m} \) such that \( [g_1, \ldots, g_{2m}] \neq 0 \).
Set \( \beta_i = g_i \) and \( \beta_{2m+1} = [g_{11}, \ldots, g_{2m}] \). Applying Theorem 4 we obtain a finite group \( \bar{H} \) with \( \mathfrak{g}_n(\bar{H}) > 1 \) and containing elements \( \tilde{g}_1, \ldots, \tilde{g}_{2m} \) with \( [\tilde{g}_1, \ldots, \tilde{g}_{2m}] \neq 0 \). Hence \( b(\bar{H}) > n^2 \).

Let \( h \in \mathfrak{g}_n(\bar{H}) \) with \( h \neq 1 \). Let \( \theta \) be an irreducible complex character of \( \bar{H} \) of degree \( > n^2 \). This exists since \( b(\bar{H}) > n^2 \). Clearly \( h \) is in the kernel of \( \theta \), that is \( \theta(h) = \theta(1) = \deg \theta \). Since \( C[H] \) is semisimple we can find an irreducible character \( \chi \) of \( \bar{H} \) with \( h \) not in the kernel of \( \chi \).
Let \( \theta \chi = \sum a_i \chi_i \) where the \( \chi_i \) are irreducible. Since \( h \in \text{kernel of } \theta \chi \) there exists a \( \chi_i \), say \( \chi_1 \) with \( h \in \text{kernel } \chi_1 \). Now \( \chi_1 \) is a constituent of \( \theta \chi \) so
\[
1 \leq [\theta \chi, \chi_1] = [\theta, \chi_1]
\]
where \( [\cdot, \cdot] \) denotes the usual inner product of characters. Hence \( \theta \) is a constituent of \( \chi_1 \). This yields
\[
n^2 < \deg \theta \leq \deg \chi_1 = (\deg \chi)(\deg \chi_1).
\]
Clearly at least one of \( \deg \chi \) or \( \deg \chi_1 \) is \( > n \) and this is the required contradiction.
By Theorem F of [3] groups $G$ with $b(G) \leq n^2$ all have abelian subgroups of index $\leq J(2n^2)$, where $J$ is the function associated with Jordan's theorem on finite linear groups. Thus $\mathfrak{R}_n(G) > 1$ is a rather restrictive condition for a group to satisfy. We discuss now a method of constructing a class of groups $G$ with $b(G) > n$ and $\mathfrak{R}_n(G) > 1$.

Let $p$ be a fixed prime and let $e \geq p$. Suppose we have $e$ groups $H_i$ each having a central subgroup $Z_i = \langle z_i \rangle$ of order $p$. Set
\[ a_i = \text{minimal degree of irreducible character } \theta_i \text{ of } H_i \text{ with } Z_i \subseteq \ker \theta_i; \]
\[ b_i = b(H_i); \]
\[ c_i = b(H_i/Z_i). \]

We suppose further that for all $i$
\[ (1) \quad (b_i/c_i) > \prod_{j=1}^e (b_j/a_j). \]

Let $U$ be an abelian group of type $(p, p)$ generated by $u, v \in U$. We define a homomorphism of
\[ Z_1 \times Z_2 \times \cdots \times Z_e \to U \]
by $z_i \mapsto uv^i$. This is clearly onto. Let the kernel be $N$. Then $N$ is a central and hence normal subgroup of $H = H_1 \times H_2 \times \cdots \times H_e$. Set $G = H/N$ so that $G \supseteq U$, a central subgroup of type $(p, p)$. Set $n+1 = \prod a_i$. We show that $v \in \mathfrak{R}_n(G)$ and that $b(G) > n$.

Let $\theta$ be an irreducible character of $G$. Then since $U$ is central $\theta|_U = (\deg \theta)\lambda$ where $\lambda$ is a linear character of $U$. Hence some subgroup of order $p$ of $U$ is the kernel of $\theta$. The subgroups of $U$ are of course $\langle v \rangle$ and $\langle uv^i \rangle$ for $i = 1, 2, \cdots, p$. Since $G$ is a homomorphic image of $H$, $\theta$ can be viewed as a character of $H$. In $H$ we write $\theta = \theta_1\theta_2 \cdots \theta_e$ where $\theta_i$ is an irreducible character of $H_i$.

Suppose first that $\langle v \rangle \subseteq \ker \theta$. Then for some $i = 1, 2, \cdots, p$ we have $\langle uv^i \rangle \subseteq \ker \theta$. Then clearly in $H$, $z_i \in \ker \theta_i$. Hence $\deg \theta_i \leq c_i$ and of course for $j \neq i$, $\deg \theta_j \leq b_j$. Thus
\[ \deg \theta \leq (\prod b_j)(c_i/b_i) < \prod a_j = n + 1. \]

Hence $\deg \theta \leq n$ and $v \in \mathfrak{R}_n(G)$. Now choose $\theta$ to be a character of $G/\langle v \rangle$ which is faithful on cyclic $U/\langle v \rangle$. Viewed in $H$ we see that for all $i$, $z_i \in \ker \theta_i$. Hence $\deg \theta_i \geq a_i$ and so $\deg \theta \geq \prod a_i = n + 1$. Therefore $b(G) > n$ and the result follows.

Using the above we can easily construct some examples.

**Example 6.** Let each $H_i$ be a nonabelian group of order $p^3$. Then
\( a_i = p, \ b_i = p \) and \( c_i = 1 \) and so equation (1) is satisfied. This yields groups \( G \) nilpotent of class 2.

Indecomposable nonnilpotent groups with nontrivial kernels can be obtained as follows.

**Example 7.** Let \( P \) denote the quaternion group of order 8 if \( p = 2 \) or the nonabelian group of order \( p^3 \) and period \( p \) if \( p \) is odd. Let \( Z \) be the center of \( P \). We have \( |Z| = p \). Let \( A \) denote the group of automorphisms of \( P \) which centralize \( Z \). \( A \) is easily seen to be isomorphic to the Symplectic group \( S_p(p) \) whose order is \( p(p + 1)(p - 1) \). Let \( \alpha \in A \) be of prime order \( q \neq p \) and let \( H \) be the semidirect product of \( P \) by the cyclic group \( \langle \alpha \rangle \). Thus \( P \) is normal in \( H \) with index \( q \) and \( Z \) is central in \( H \).

Let \( \chi \) be an irreducible character of \( H \). By Proposition 1.2 of [2], either \( \chi|P \) is irreducible or \( \chi|P \) is the sum of \( q \) conjugate characters under the action of \( \langle \alpha \rangle \). In the first case \( \deg \chi = 1 \) or \( p \). In the second case let \( \phi \) be an irreducible constituent of \( \chi|P \). If \( \deg \phi = p \) then \( \phi \) vanishes off \( Z \). Since \( \alpha \) centralizes \( Z \), \( \phi^\alpha = \phi \), a contradiction. Hence \( \deg \phi = 1 \) and \( \deg \chi = q \). Thus \( H \) has characters of degree 1, \( p \) and \( q \) only. Moreover if \( H = H_i \) we have easily \( a_i = p, \ b_i = \max(p, q) \) and \( c_i = q \).

If \( p = 2 \) choose \( q = 3 \). Then \( b(H) = 3 \) and \( Z \subseteq \mathfrak{Z}(H) \). If \( p > 2 \) then choose \( q \) to divide \( p(p + 1)(p - 1) \) so \( p > q \). Hence in this case equation (1) is satisfied and the group \( G \) constructed has the required property.

**References**


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