EXISTENCE OF INVARIANT MEASURES
FOR MARKOV PROCESSES. II

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The purpose of this note is to improve the results obtained in [2] by relaxing the assumptions on the processes under consideration. Thus we impose weaker topological structure on the states space and, what is more important, the continuity of \( P(x, A) \), when \( A \) is open, is not assumed.

1. Notation. We shall use the notation of [1] for topological and measure theoretic concepts.

Let \( X \) be a normal topological space. Let \( P(x, A) \) be the transition probabilities of a Markov Process:

1.1. For a fixed \( x \in X \) the set function \( P(x, \cdot) \) is a measure, on the Borel sets, of total measure one.

1.2. For a fixed Borel set \( A \), the function \( P(\cdot, A) \) is Borel measurable.

By a measure we shall mean a countably additive positive measure, unless otherwise stated. Let us denote by \( \mathfrak{r b a} \) the set of regular bounded finitely additive signed measures on \( X \) and by \( \mathfrak{r c a} \) those elements of \( \mathfrak{r b a} \) which are countably additive. The transition probabilities induce an operator on the bounded measurable functions by

1.3. \( (Pf)(x) = \int f(y) P(x, dy) \).

Also if \( \mu \) is a bounded finitely additive signed measure one defines

1.4. \( (\mu P)(A) = \int P(x, A) \mu(dx) \).

It is well known that

1.5. \( \int (Pf)(x) \mu(dx) = \int f(x) (\mu P)(dx) \)

and that \( \mu P \) is countably additive if \( \mu \) is.

Throughout the paper we assume:

1.6. If \( f \in C(X) \) then \( Pf \in C(X) \), where \( C(X) \) denotes the continuous functions. Also:

1.7. If \( \mu \in \mathfrak{r c a} \) then \( \mu P \in \mathfrak{r c a} \).

These two conditions are always satisfied under the assumptions of [2]: under the assumptions of [2] every countably additive measure is regular. Another example is given by \( (Pf)(x) = f(\phi(x)) \) where \( \phi \) is a homeomorphism of \( X \) onto \( X \).

Received by the editors May 16, 1965

1 The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research, under Grant AF EOAR 65-27 through the European Office of Aerospace Research (OAR) United States Air Force.
2. Invariant measures. Let $P^*$ be the adjoint operators of $P$ on $C(X)$. Thus by Theorem IV.6.2 of [1] $P^*$ is defined on $r$ b $a$ and from 1.5 and 1.7 follows that:

2.1. If $\mu \in r$ c $a$ then $P^*\mu = \mu P$.

Let $\mu$ be a positive finitely additive measure. Then $\mu = \mu_1 + \mu_2$ where $\mu_1$ is countably additive and $\mu_2$ is purely finitely additive i.e.: if $\mu_2 \geq \lambda \geq 0$ and $\lambda$ is countably additive then $\lambda = 0$. Also $\mu_1 \geq 0$, $\mu_2 \geq 0$.

This decomposition and its uniqueness are proved in [3, p. 52]. Clearly if $\mu$ is regular so are $\mu_1$ and $\mu_2$.

**Lemma 1.** Let $\mu \in r$ b $a$ and $\mu \geq 0$. If $P^*\mu = \mu$ then $P^*\mu_1 = \mu_1$.

**Proof.** Since $\mu_1 + \mu_2 = P^*\mu_1 + P^*\mu_2$ and $P^*\mu_1$ is countably additive it follows that $\mu_1 \geq P^*\mu_1$: let $P^*\mu_2 = \sigma_1 + \sigma_2$ where $\sigma_1$ is countably additive and $\sigma_2$ purely finitely additive then $\mu_1 = P^*\mu_1 + \sigma_1$. But

$$\mu_1(X) = \int d\mu_1 = \int P1d\mu_1 = (P^*\mu_1)(X),$$

where 1 is the function identically to one and by 1.1 $P1 = 1$. Thus $0 \leq (\mu_1 - P^*\mu_1)(A) \leq (\mu_1 - P^*\mu_1)(X) = 0$.

For any $0 \leq \mu \in r$ b $a$ put

$$\mu_n = \mu + P^*\mu + \cdots + P^{n-1}\mu.$$

**Theorem 1.** Let $A$ be a fixed compact set. Then either

(a) for every $0 \leq \mu \in r$ b $a$ $\lim \mu_n(A) = 0$, or

(b) there exists a measure $0 \leq \mu \in r$ c $a$ with $\mu = \mu P$ and $\mu(A) \neq 0$.

**Proof.** Let $0 \leq \mu \in r$ c $a$ be such that $\mu_n(A) \geq \delta > 0$ for a subsequence, $n_i$, of the integers. Let $\mu$ be a weak star limit of $\mu_{n_i}$. Clearly $P^*\mu = \mu$. If $B$ is any open set containing $A$ let $f \in C(X)$ satisfy $0 \leq f \leq 1$, $f(X - B) = 0$, $f(A) = 1$. Then

$$\mu(B) \geq \int f \, d\mu = \lim \int f \, d\mu_{n_i} \geq \delta.$$

Since the measure $\mu$ is regular also $\mu(A) \geq \delta$. Finally let $\mu = \mu_1 + \mu_2$ be the decomposition of Lemma 1. The theorem will be proved if we show that $\mu_2(A) = 0$ since then $\mu_1$ will satisfy (b). But the restriction of $\mu_2$ to $A$ is countably additive, by Theorem III.5.13 of [1] and thus is zero.

Given an invariant measure $0 \leq \mu \in r$ c $a$ if $\int fd\mu = 0$ where $0 \leq \mu \in C(X)$, then $\int Pf \, d\mu = 0$ too, hence $\int (\sum_{n=1}^{\infty} P^nf) \, d\mu = 0$. If $P$ is such that whenever $0 \leq f \in C(X)$ and $f \neq 0 \sum P^nf > 0$ then $\mu$ never vanishes.
on open sets, or the kernel of $\mu$ is all of $X$.

Let $K(\mu)$ be the kernel of $0 \leq \mu \leq \mu_{c}$ i.e.: if $x \in K(\mu)$ and $N$ is a neighborhood of $x$ then $\mu(N) \neq 0$.

**Theorem 2.** Let $0 \leq \mu \leq \mu_{c}$ be invariant. Then

$$P^{n}(x, K(\mu)) = 1, \quad x \in K(\mu), \quad n = 1, 2, \cdots$$

**Proof.** Let us show (2.3) for $n=1$. Take a fixed $x \in K(\mu)$. The measure $P(x, \cdot) = \delta_{x}P$ is regular by 1.7. Thus it is enough to show that if $A$ is a closed set disjoint to $K(\mu)$ then $P(x, A) = 0$. Let $f \in C(X)$ be such that $0 \leq f \leq 1$, $f(K(\mu)) = 0$ and $f(A) = 1$. Then

$$0 = \int f \, d\mu = \int (Pf) \, d\mu.$$

Thus $Pf = 0$ a.e. and since $Pf$ is continuous and $x \in K(\mu)$ $(Pf)(x) = 0$.

Finally $P(x, A) \leq (Pf)(x)$ since $f(A) = 1$.

Following [2] let us define:

**Definition.** A set $A \subset X$ is called self-contained if $P(x, A) = 1$ for all $x \in A$.

Also put for a self contained set $A$

$$A^{n} = \{x: P^{n}(x, A) > 0\}, \quad A^{*} = \bigcup_{n=1}^{\infty} A^{n} - A.$$

Then if $\mu$ is an invariant measure

$$\mu(A^{*}) = 0$$

and also if $A$ is self contained so is $X - A^{*} - A$.

These facts are proved in Theorem 4 and Lemma 5 of [2], respectively, and the proof is valid in our case too.

It should be noted that even when $A$ is closed $A \cup A^{*}$ does not have to be open: consider the identity transformation. Thus we can not continue to prove results obtained in Theorem 6 and 7 of [2]. Clearly the proof of Theorem 9 fails in our case.

**Bibliography**


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